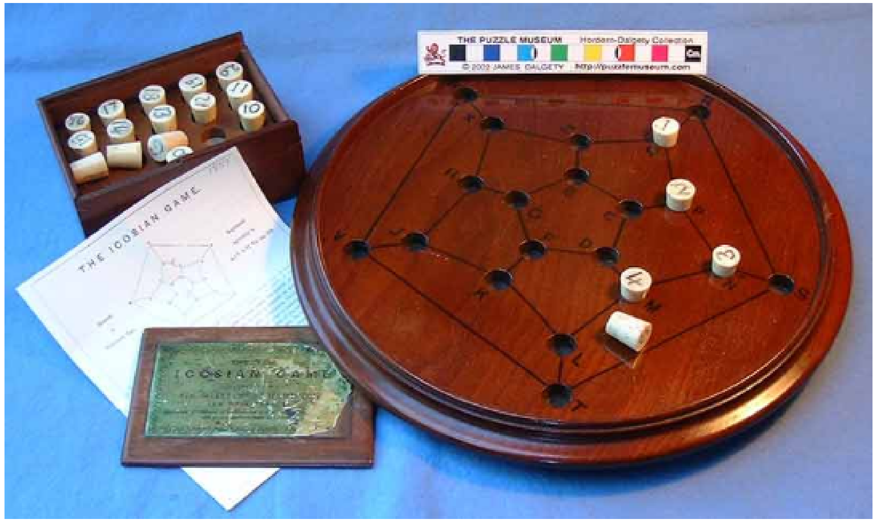
On Hamilton cycles in highly symmetric graphs

Torsten Mütze
University of Warwick

Slovenian Conference on Graph Theory 2023

Introduction

- Sir Williams Rowan Hamilton (1805-1865): **Icosian game**



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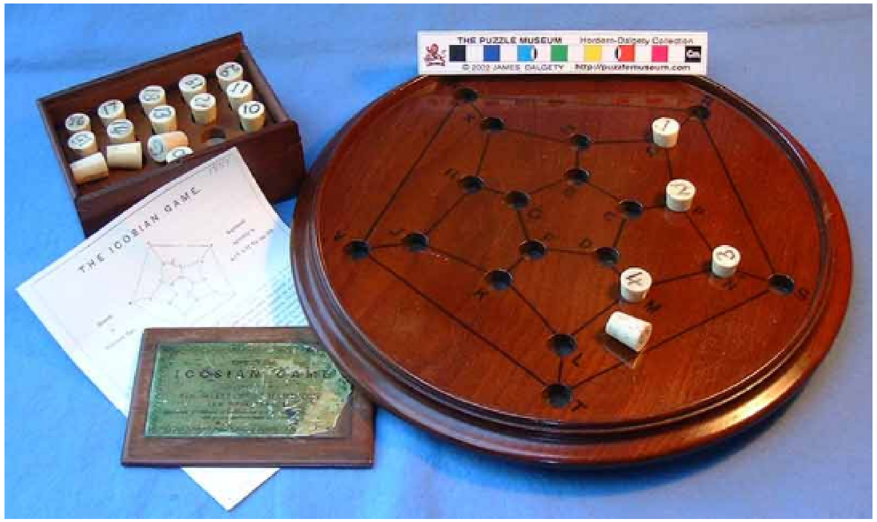

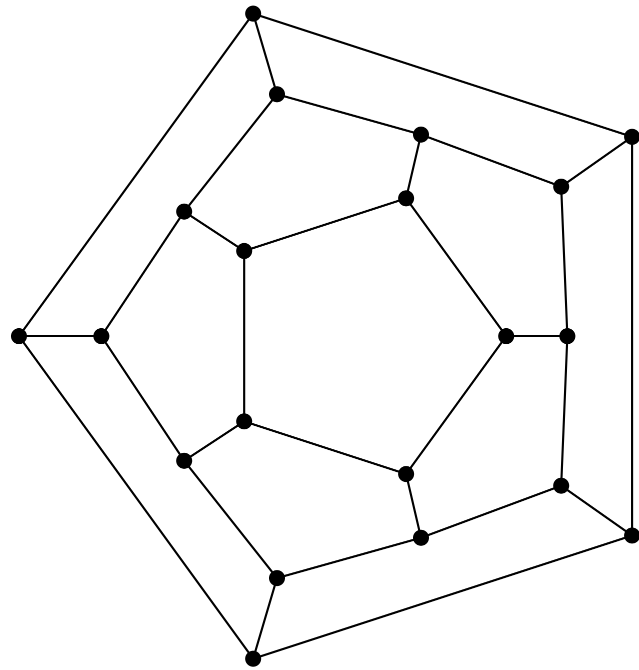

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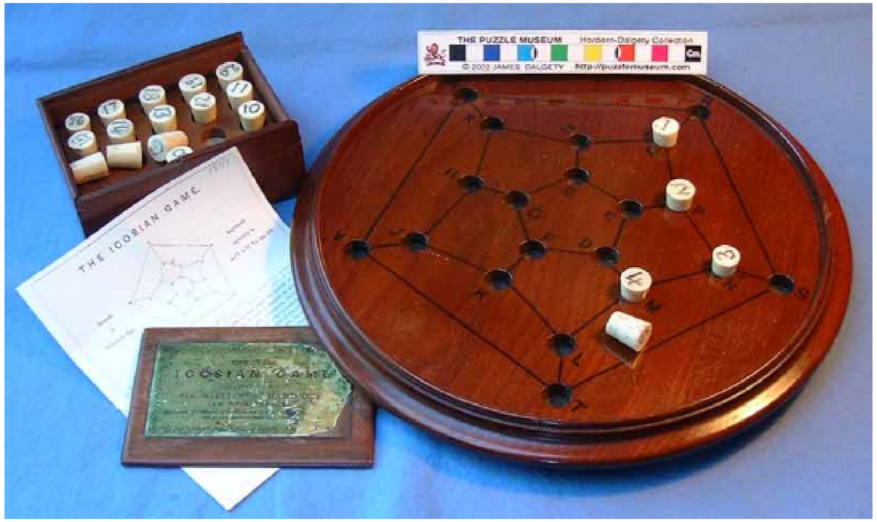

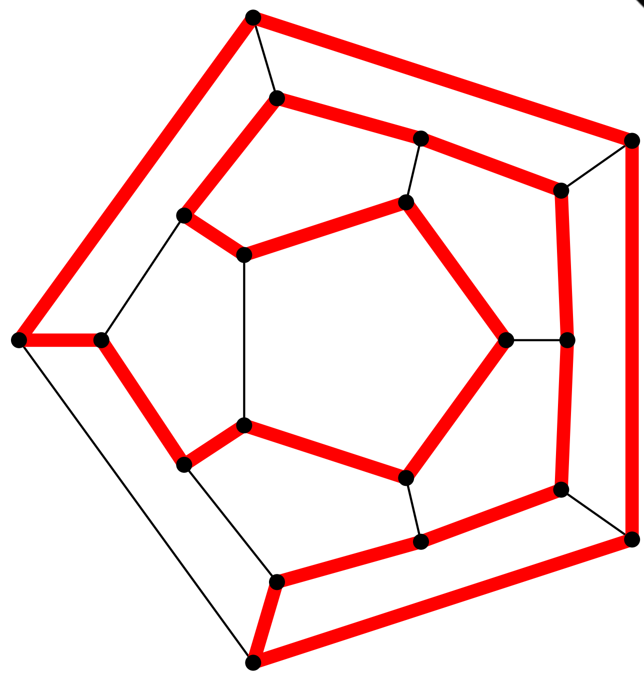

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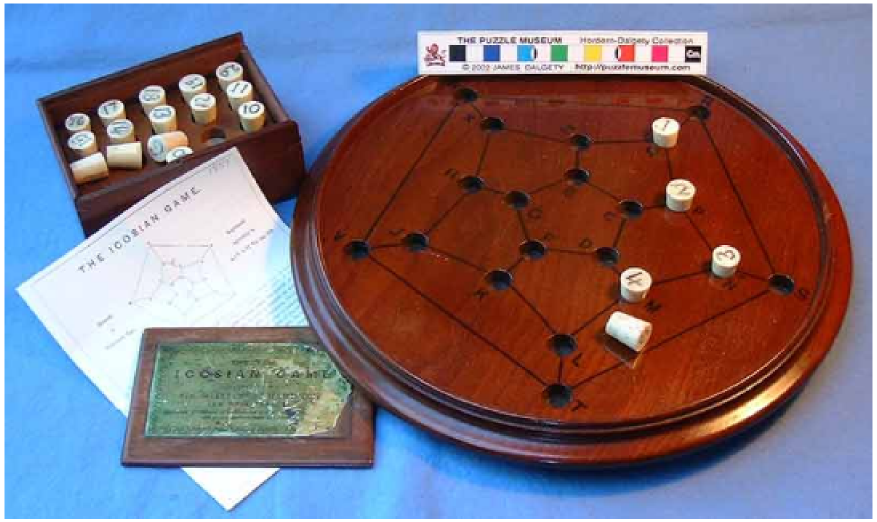

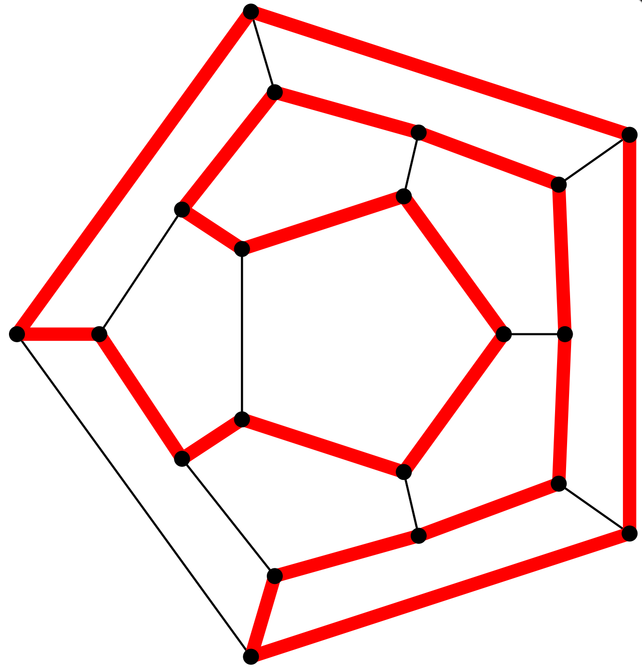

- **Definition:** A **Hamilton path/cycle** in a graph
  is a path/cycle in a graph that visits every vertex exactly once

Hamilton cycle problem

- **Problem:** Does a graph have a Hamilton path/cyle?

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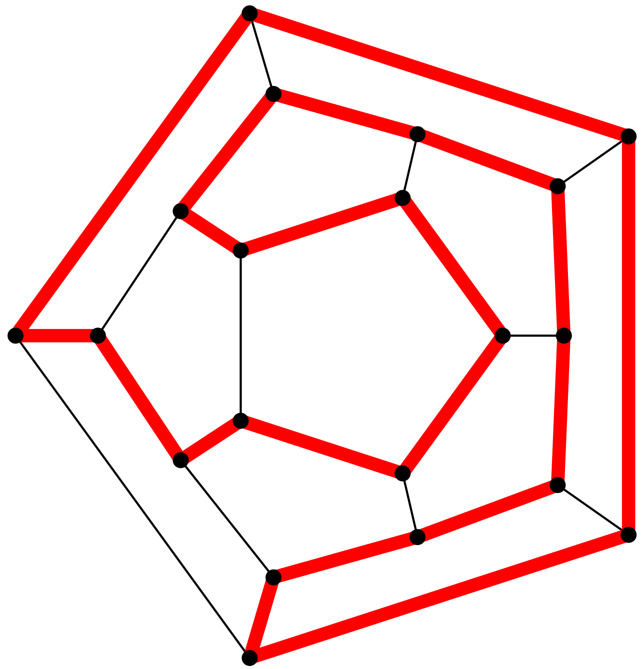
- optimization (TSP, approximation) [Christofides 1976], [Karlin, Klein, Garan 2021], [Svensson, Tarnawski, Végh 2018], [Zenklusen 2018]

Lovász' conjecture

- the dodecahedron is **vertex-transitive**;



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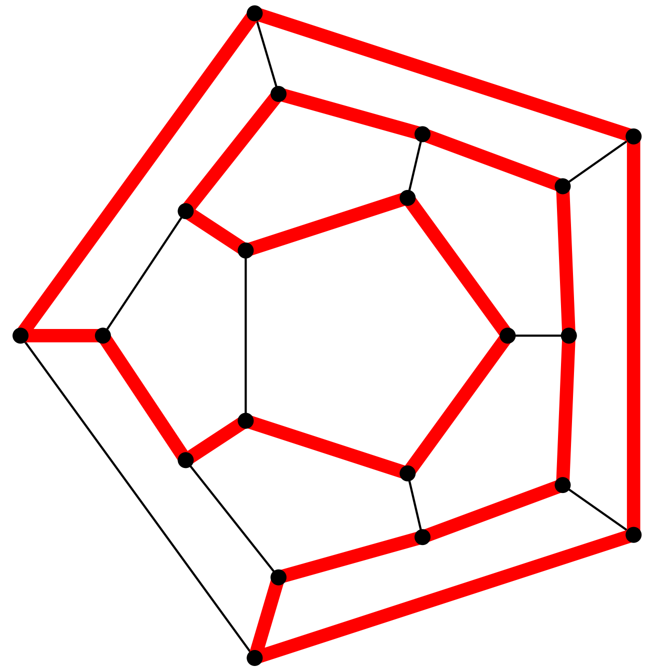

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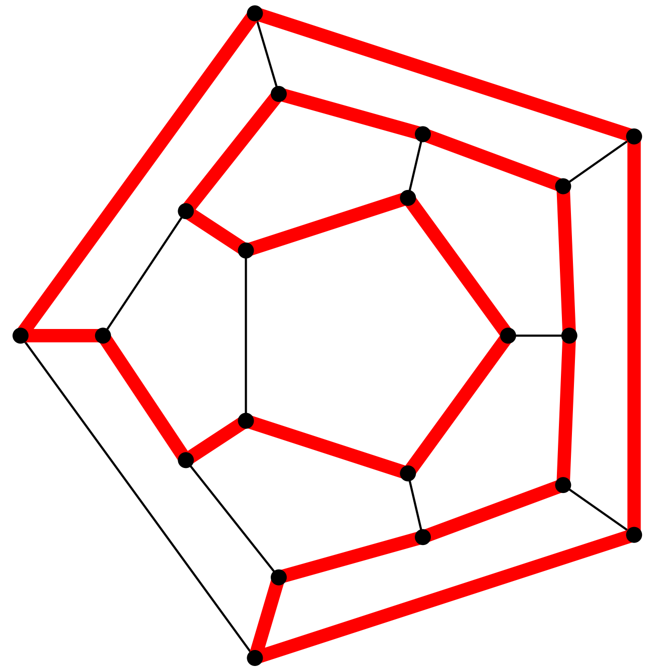

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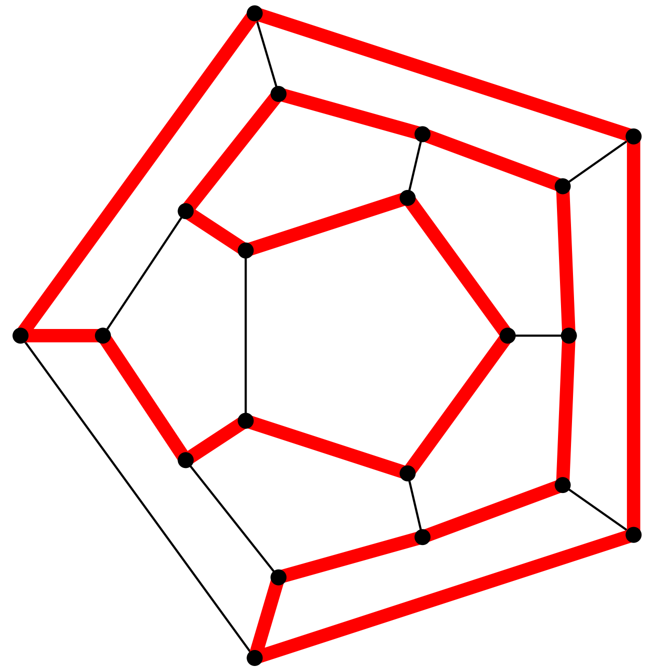

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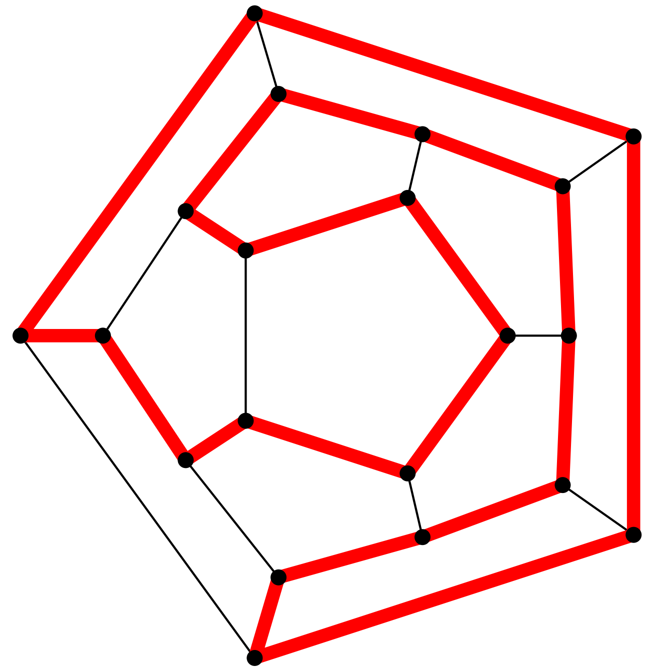

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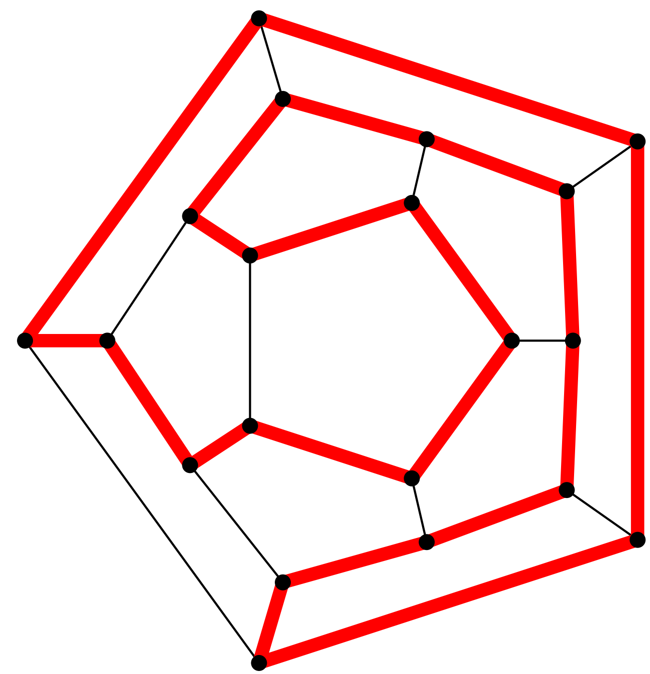

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- **Conjecture** [Babai 1995]:
  There is $\varepsilon > 0$ and infinitely many connected vertex-transitive graphs with longest cycle $(1 - \varepsilon)n$.

Lovász' conjecture

- proved for special cases:
  - $n = p, n = 2p, n = 3p, n = 4p$ (p prime)
    $n = p^2, n = p^3, n = p^4$
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- Cayley graphs
  - abelian groups
  - finite p-groups [Witte 1986]
  - symmetric group + transpositions [Tchuente 1982]

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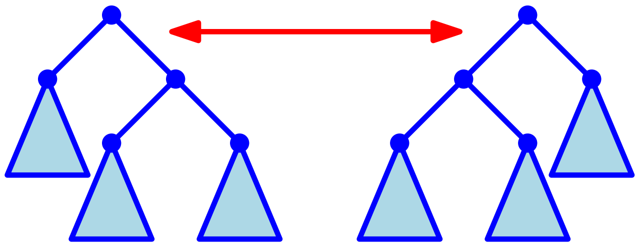

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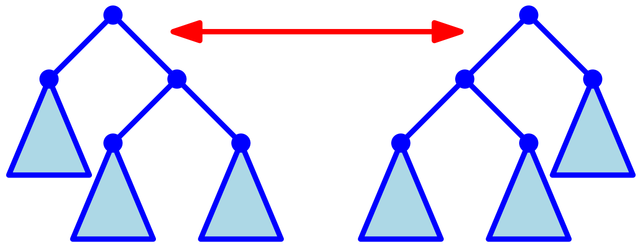

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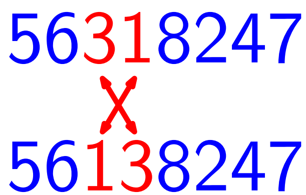

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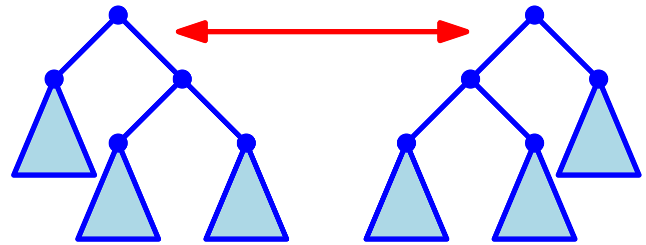

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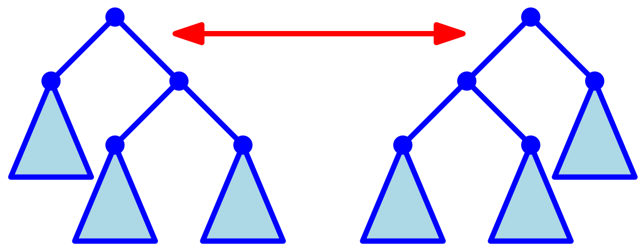

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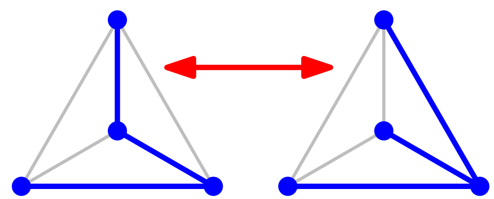
  - spanning trees by **edge exchanges**
    [Cummins 1966], [Holzmann, Harary 1972]



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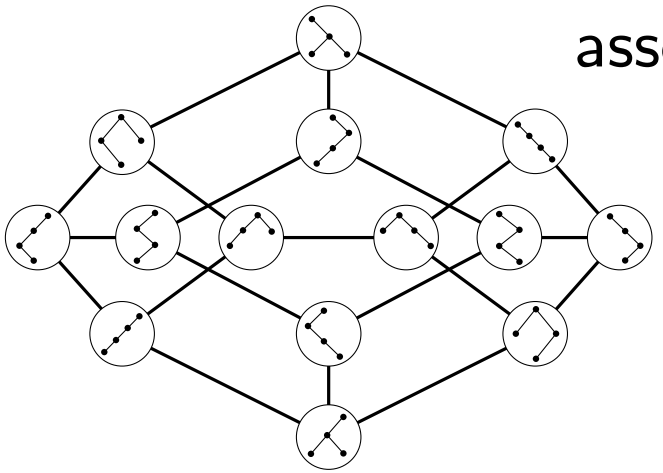

associahedron

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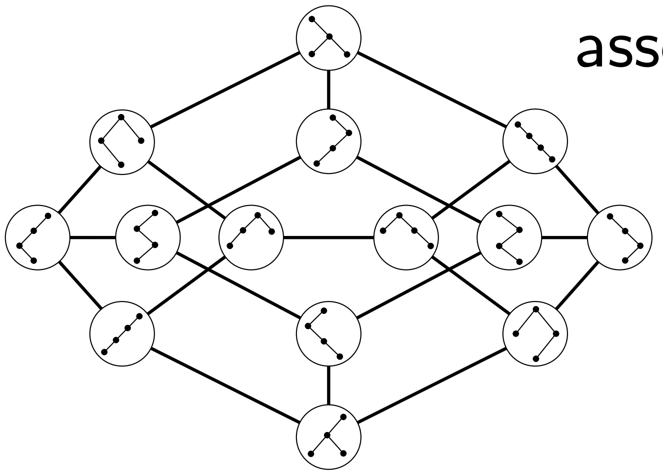

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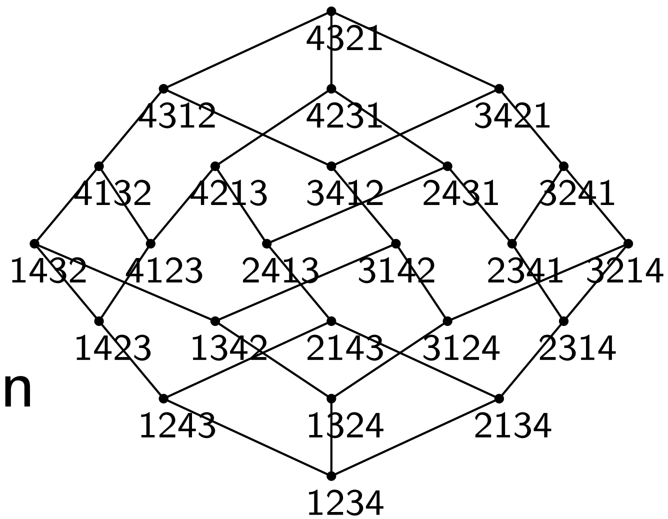

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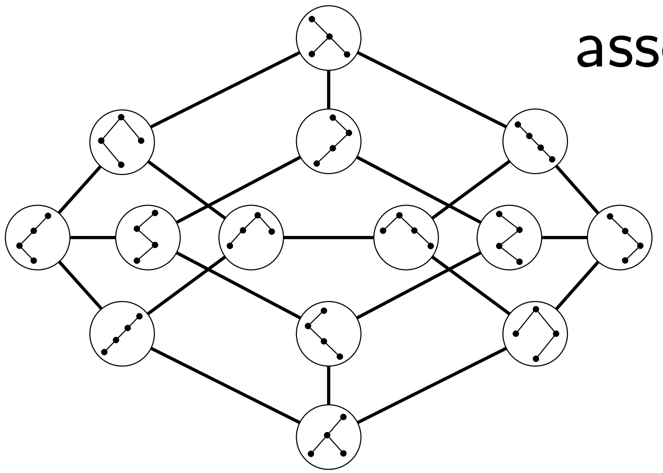

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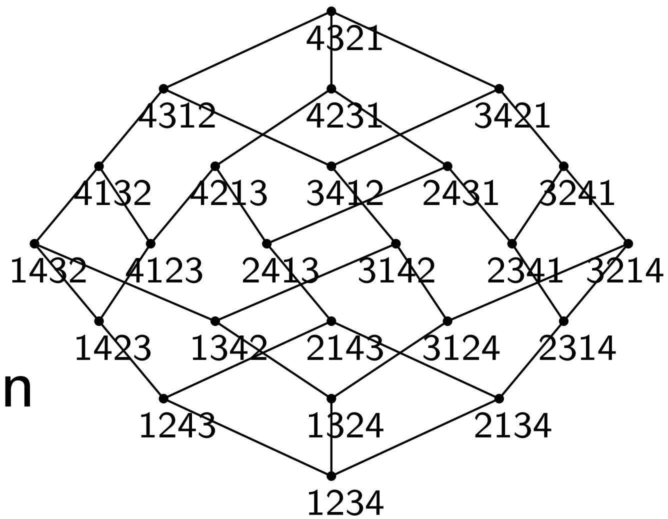

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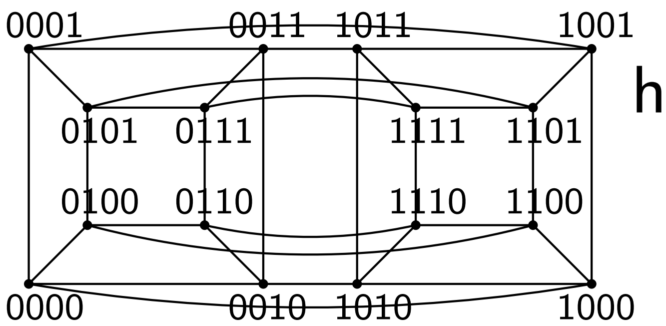

hypercube

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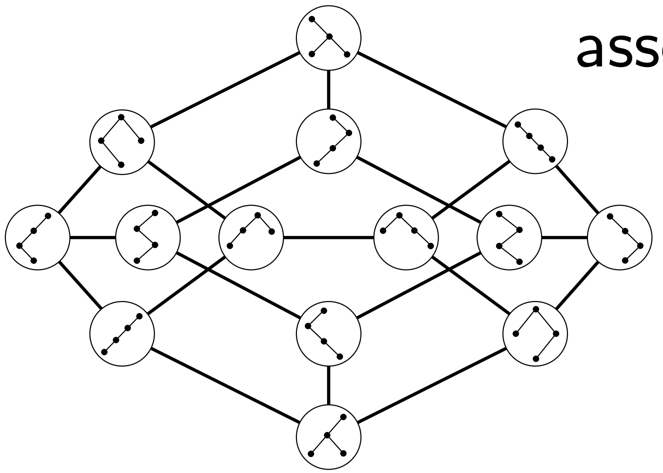

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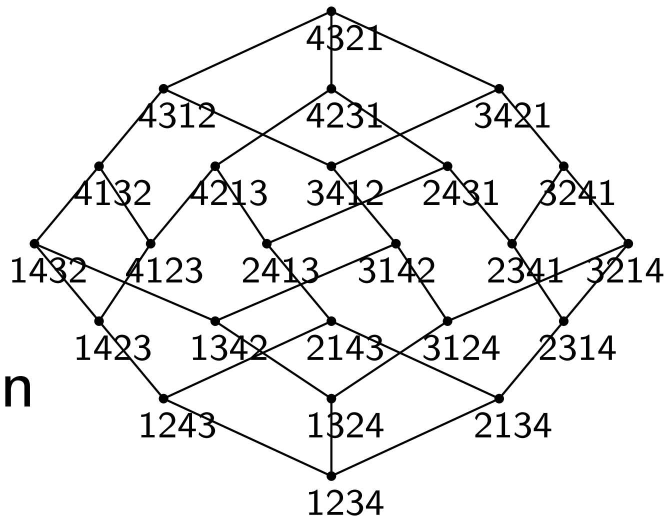

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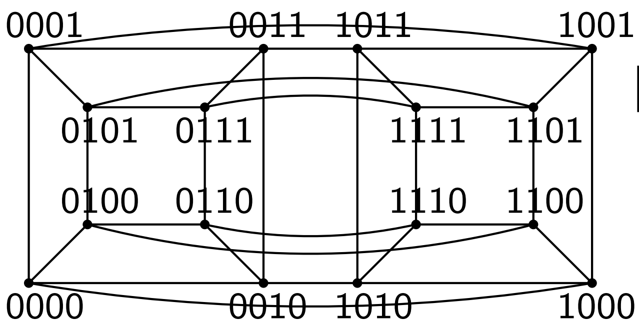

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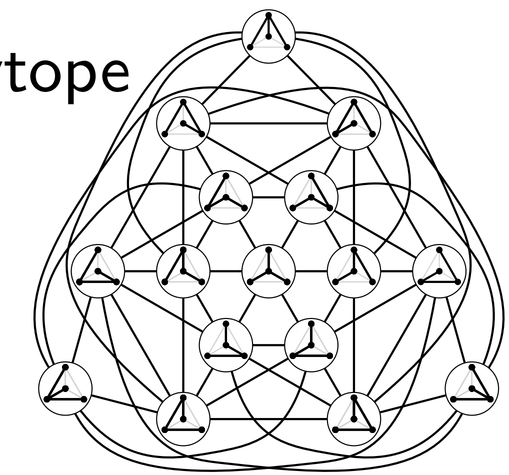

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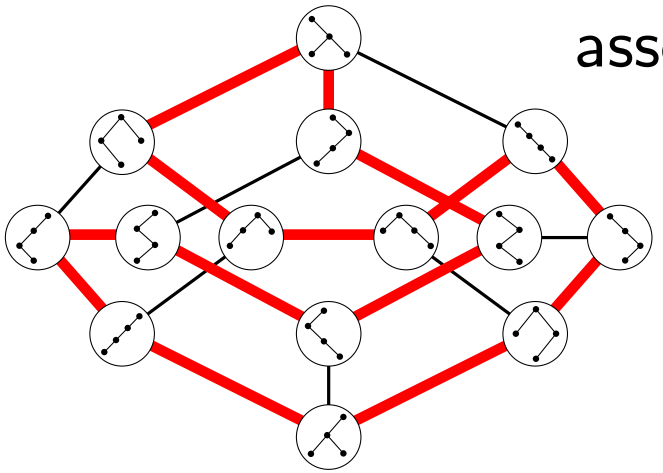

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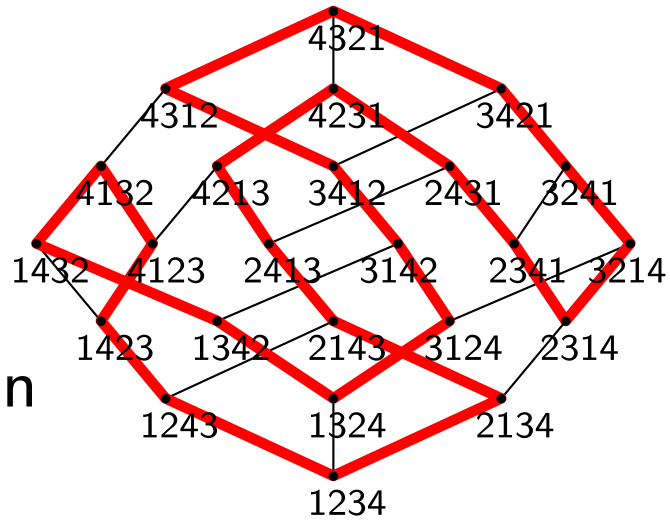

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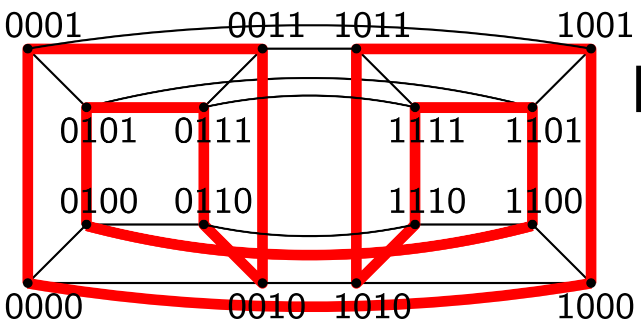

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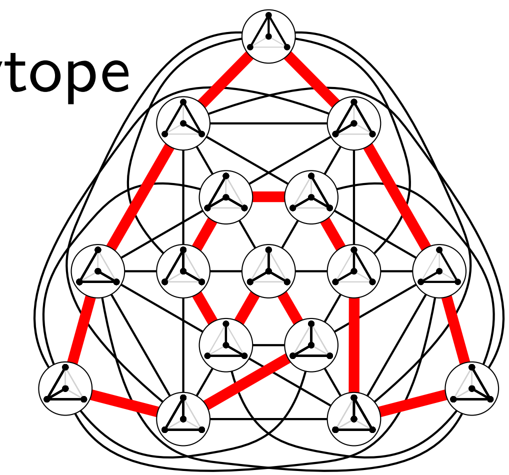

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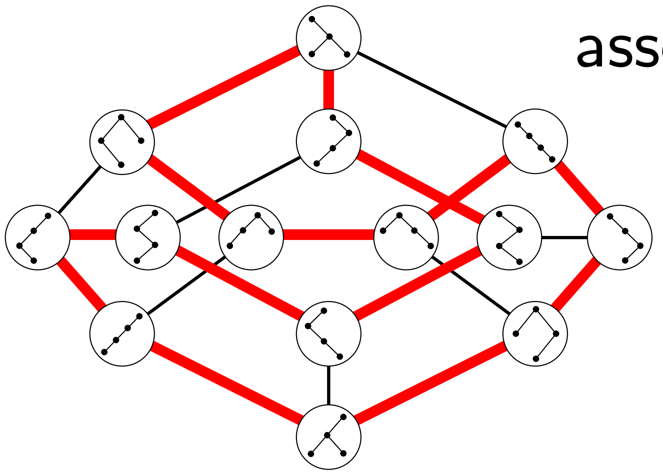

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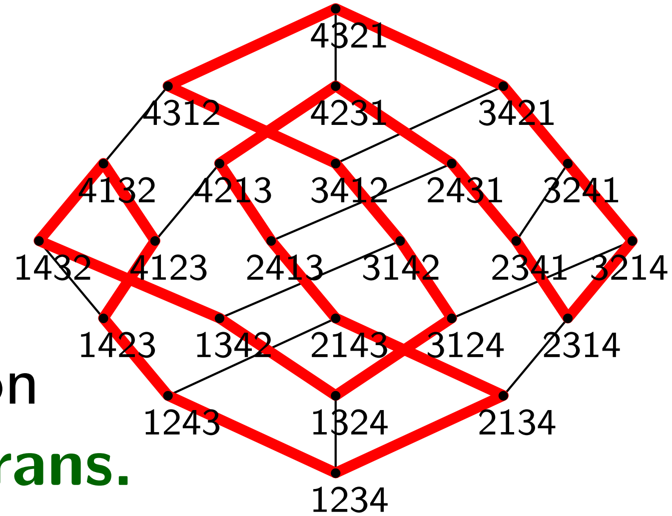

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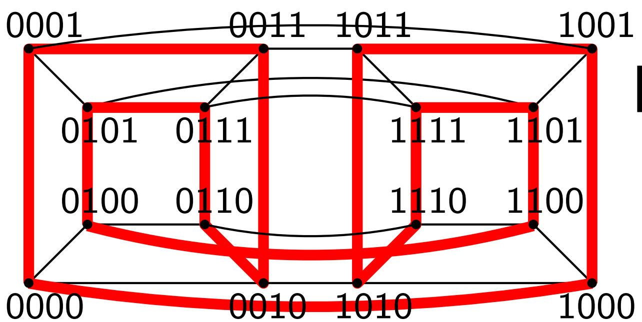

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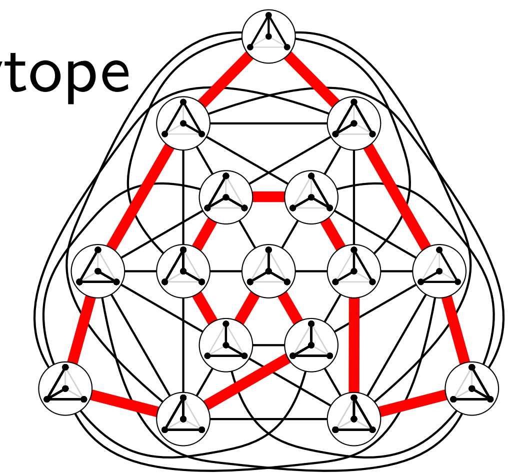

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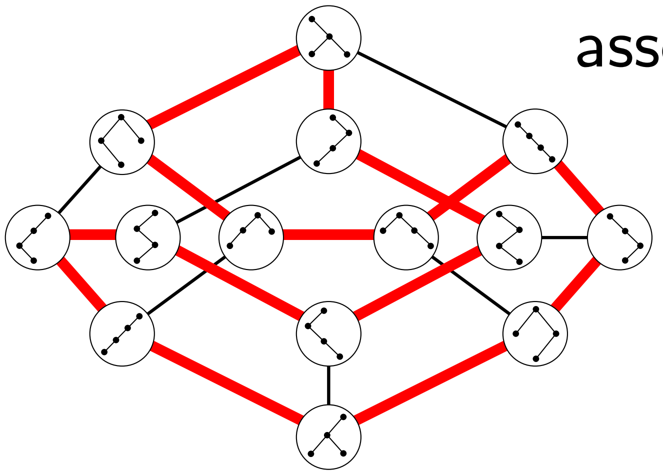

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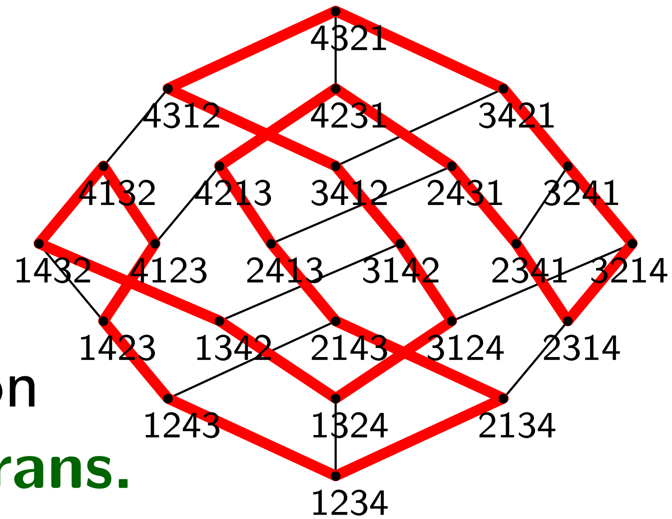

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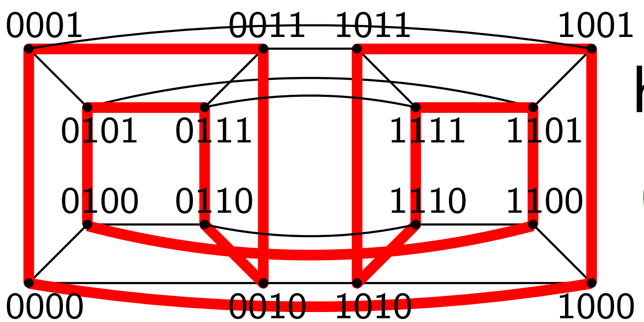

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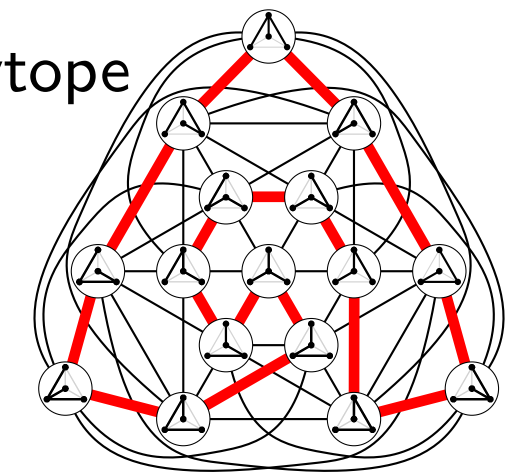

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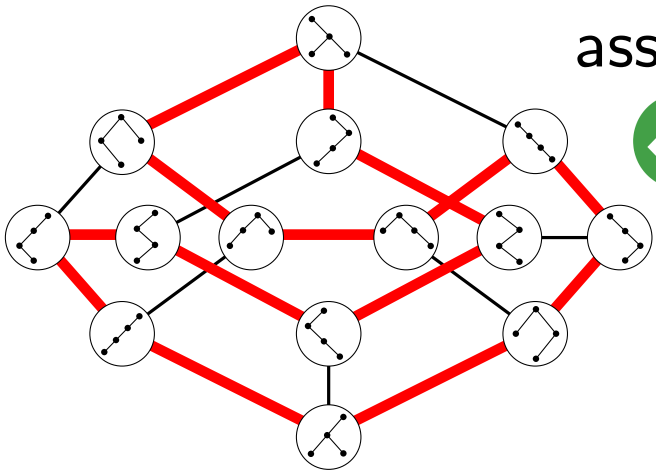
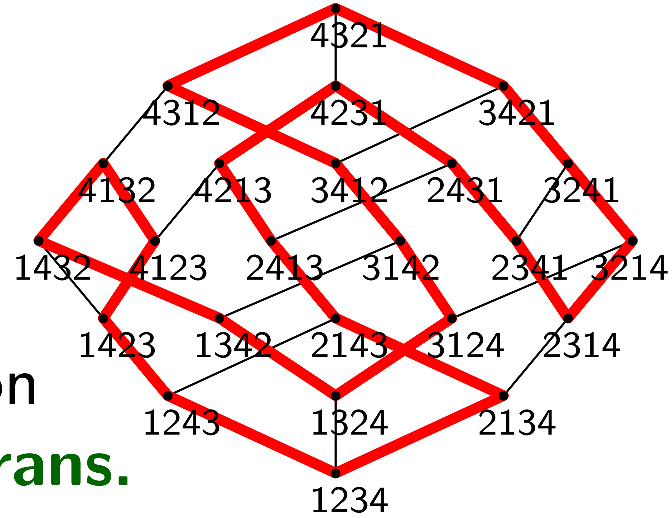


associahedron

regular



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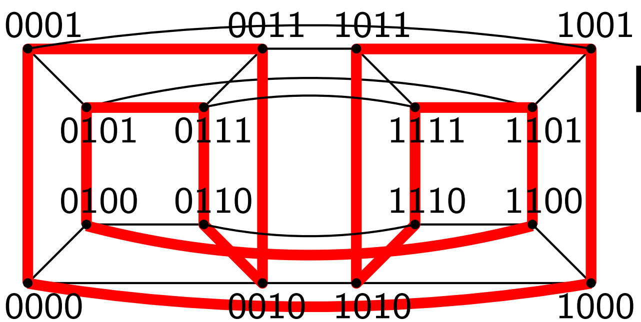

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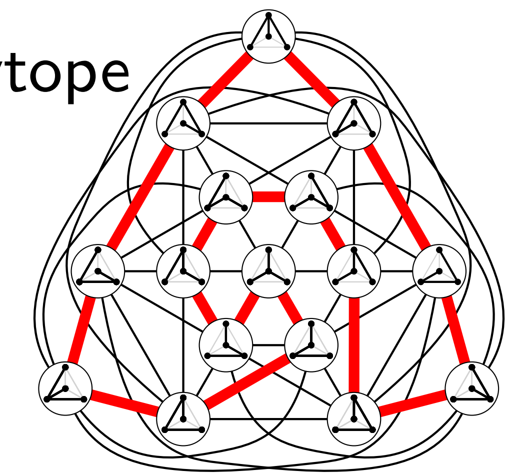

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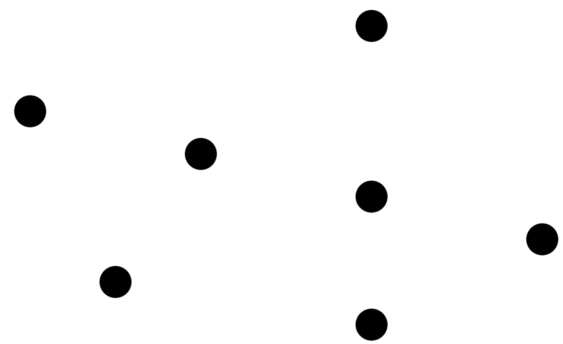

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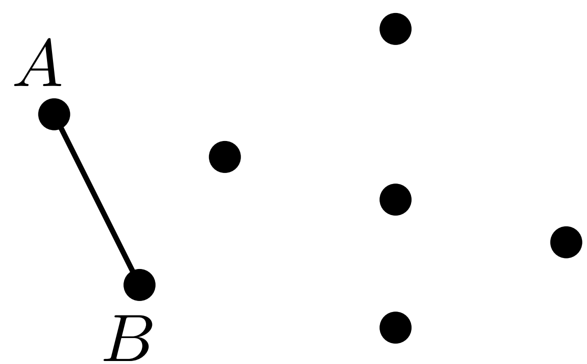
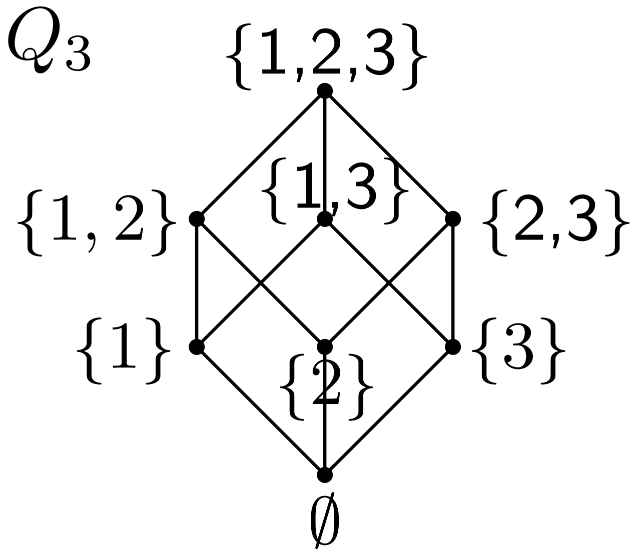
edges = pairs of sets (A, B) satisfying conditions on the intersection of A and B



Hypercube

- **Hypercube** Q_n

  vertices = **all** subsets of $[n] = \{1, 2, \ldots, n\}$
  edges = pairs (A, B) with $|A \triangle B| = 1$



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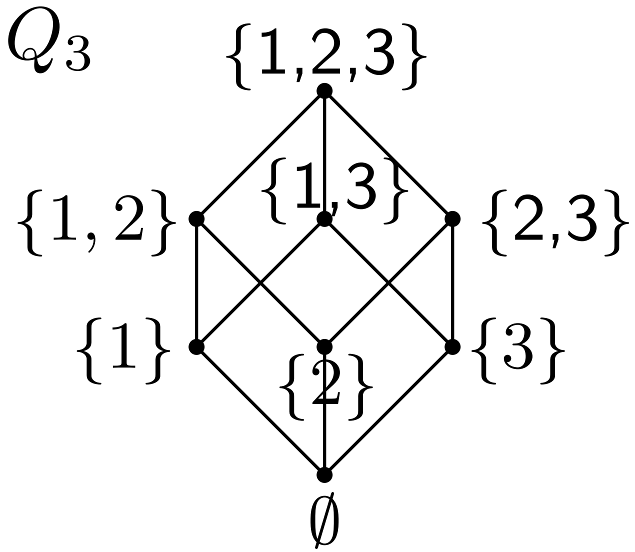

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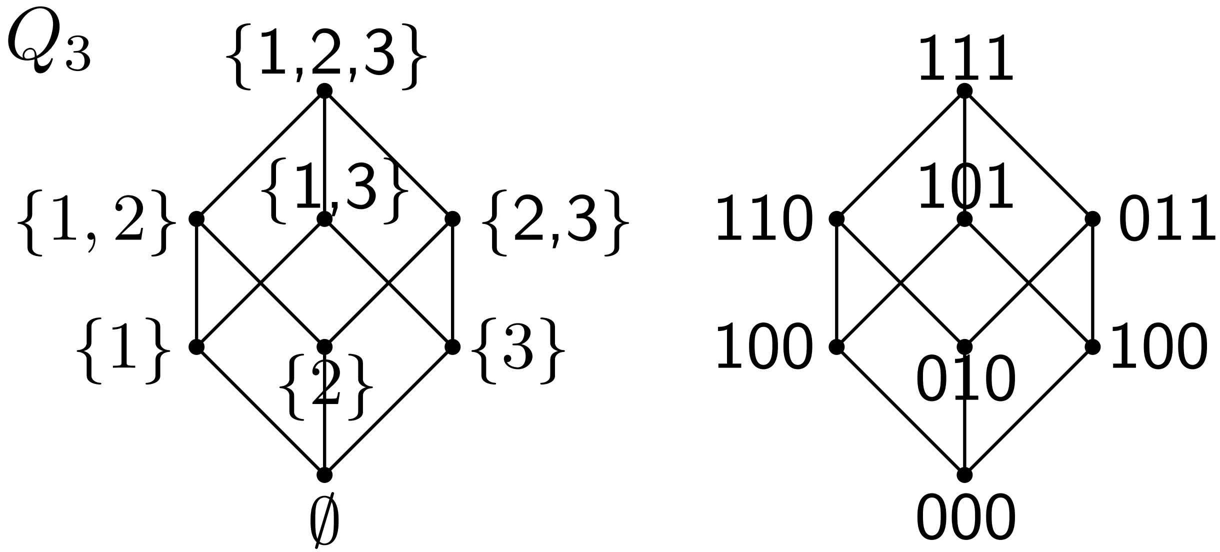

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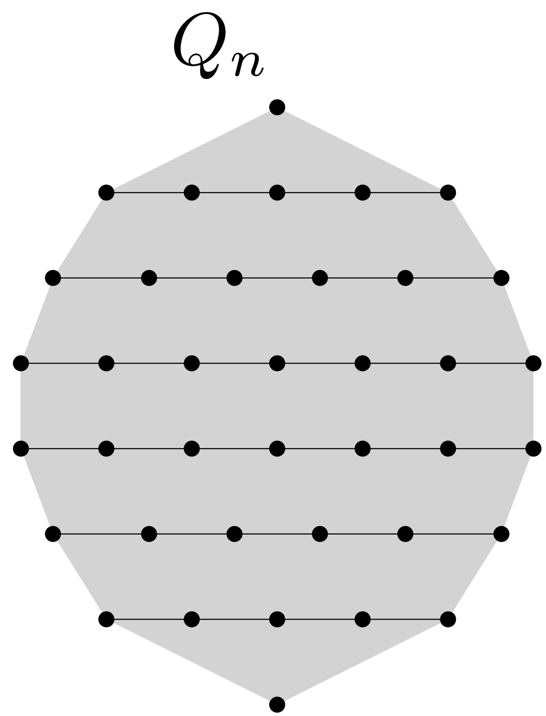

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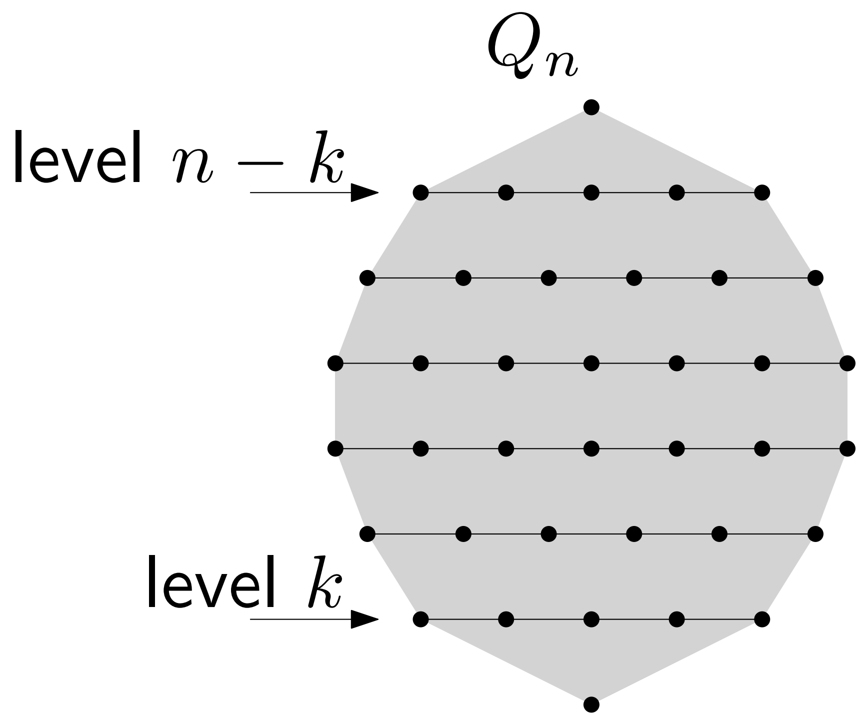

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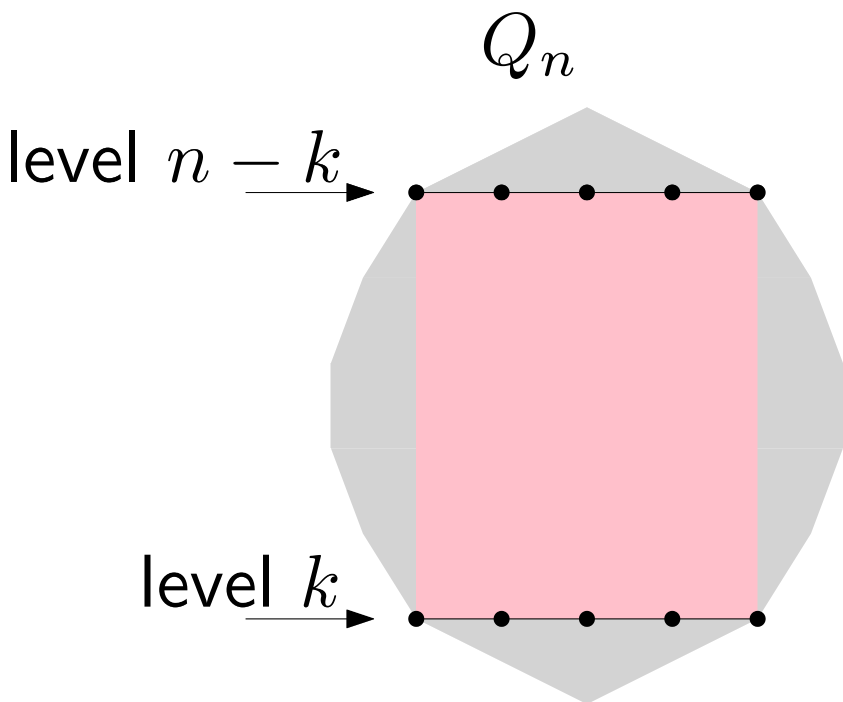

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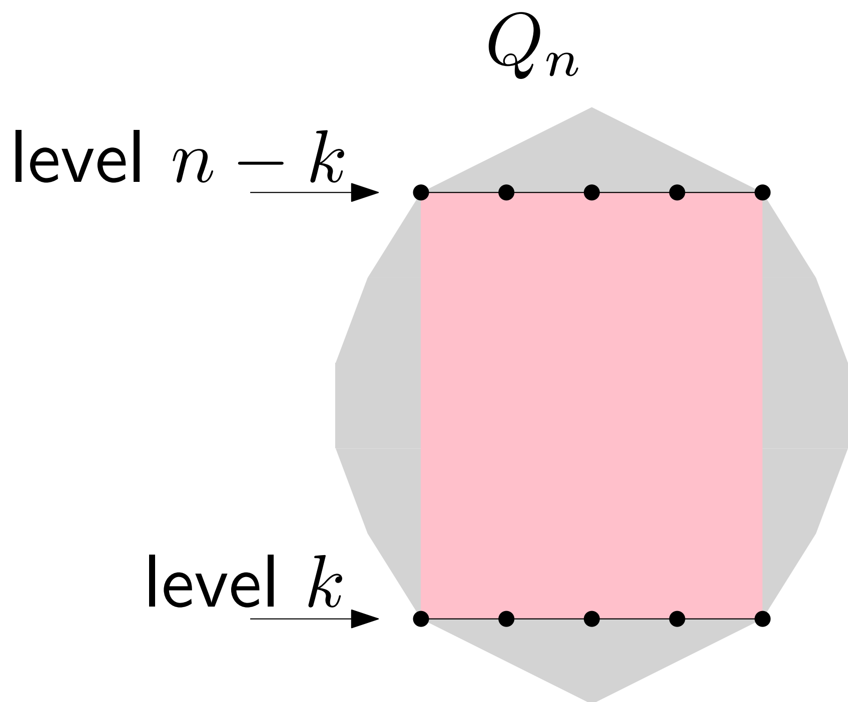

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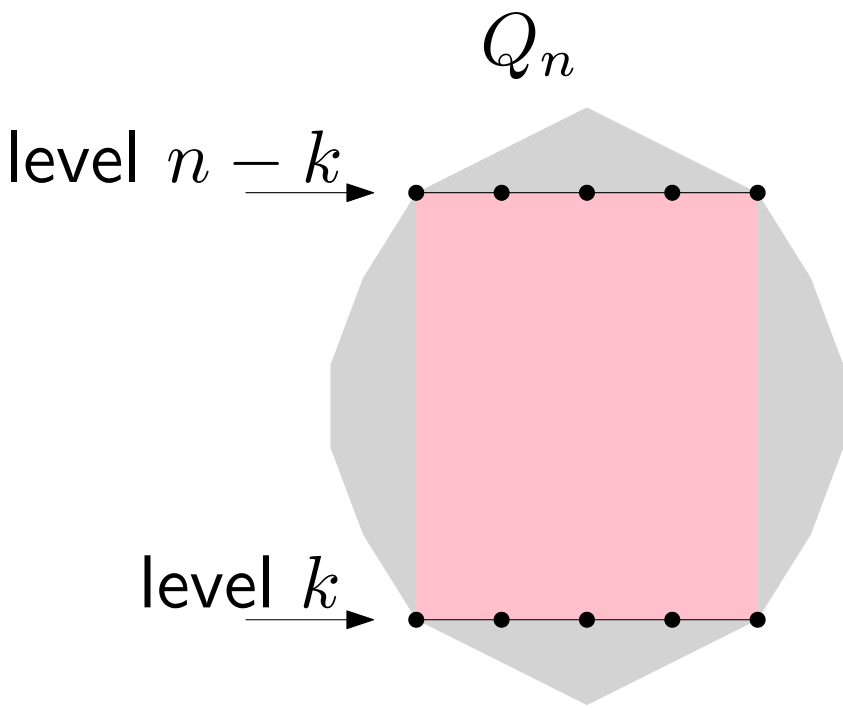

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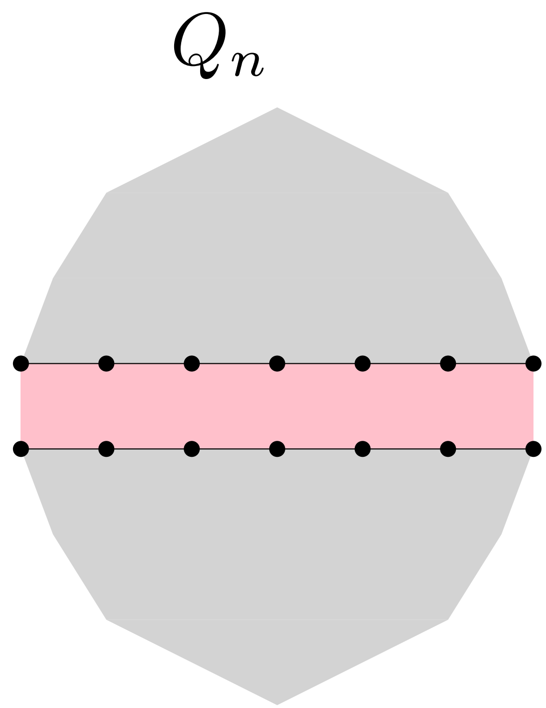

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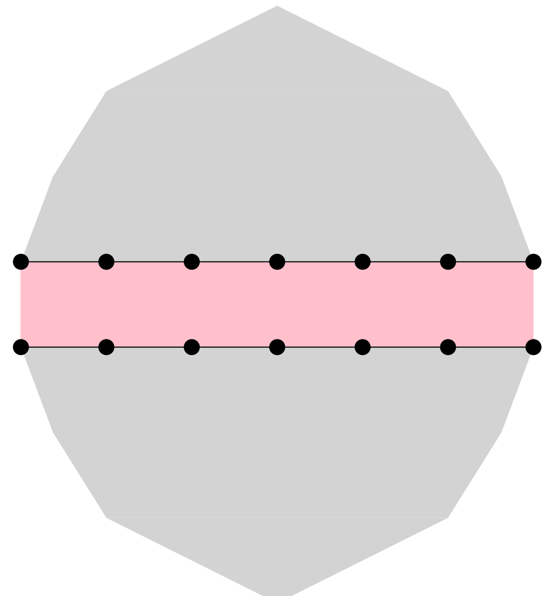

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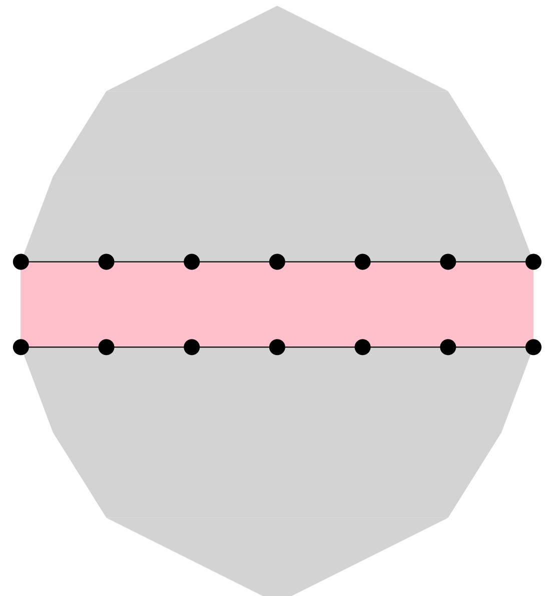

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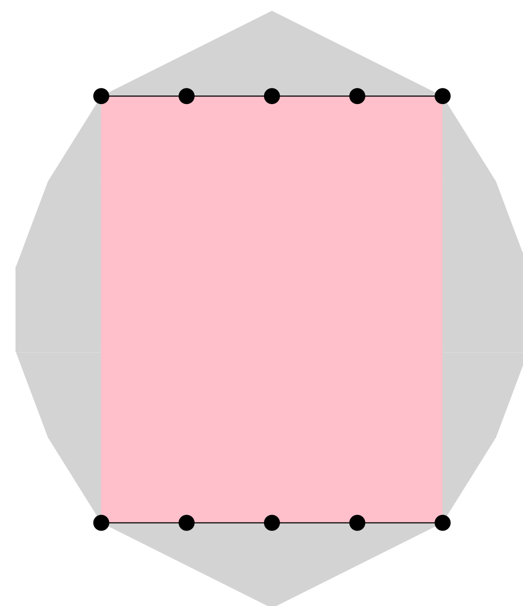

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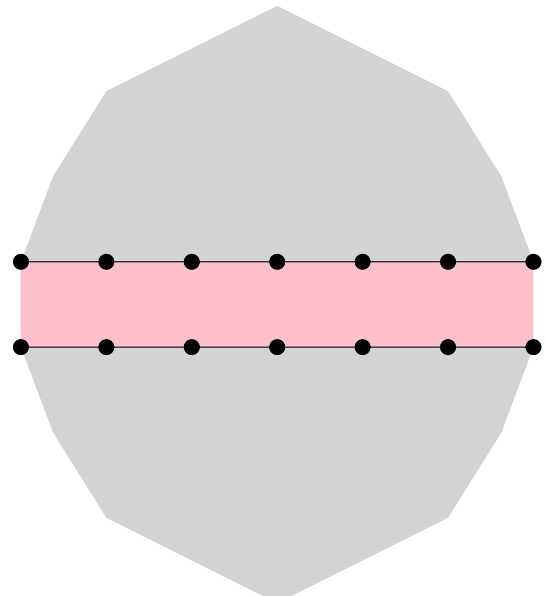

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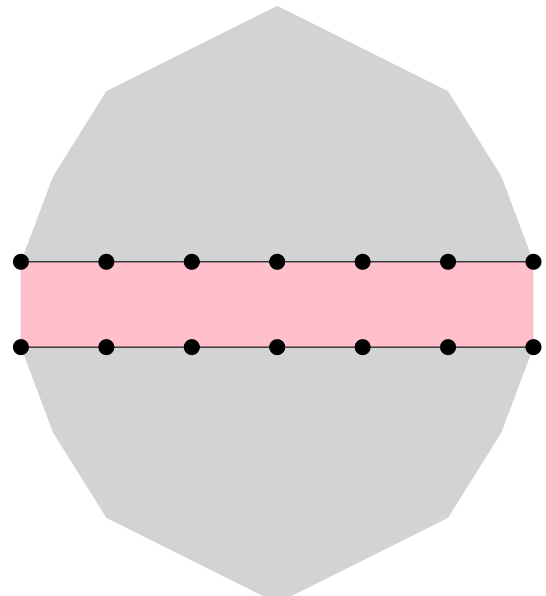

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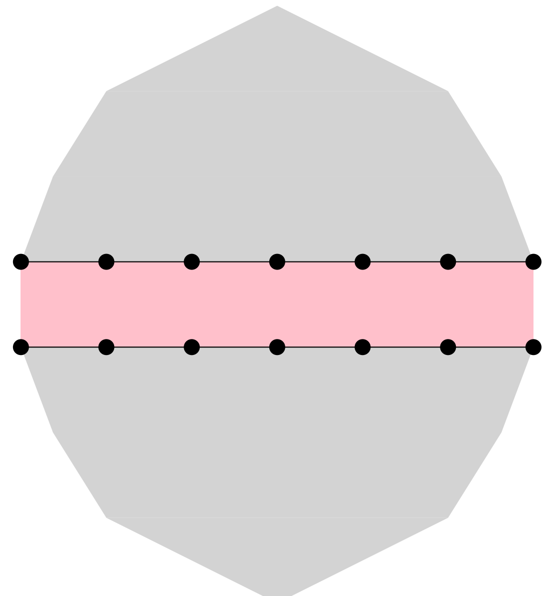

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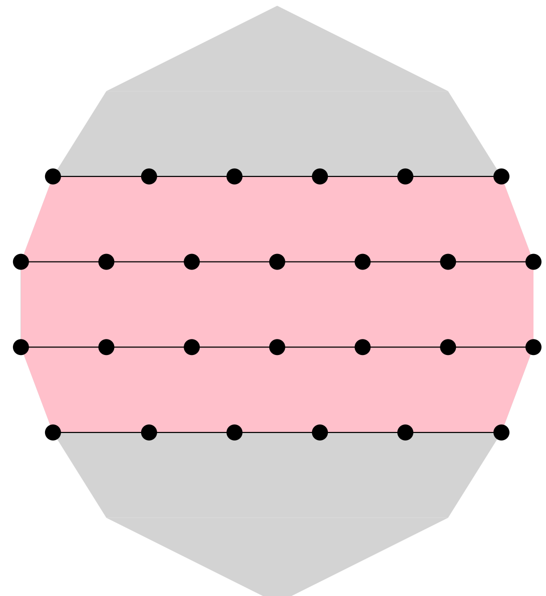

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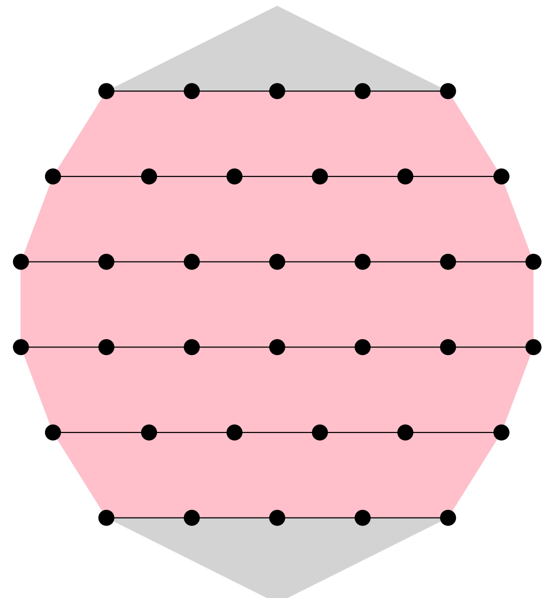

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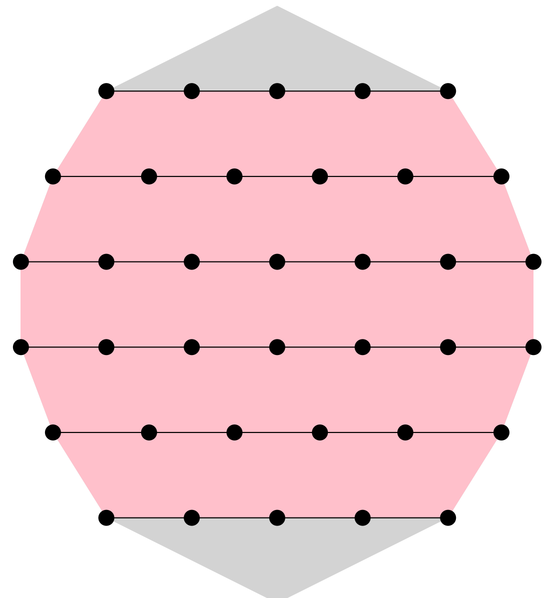

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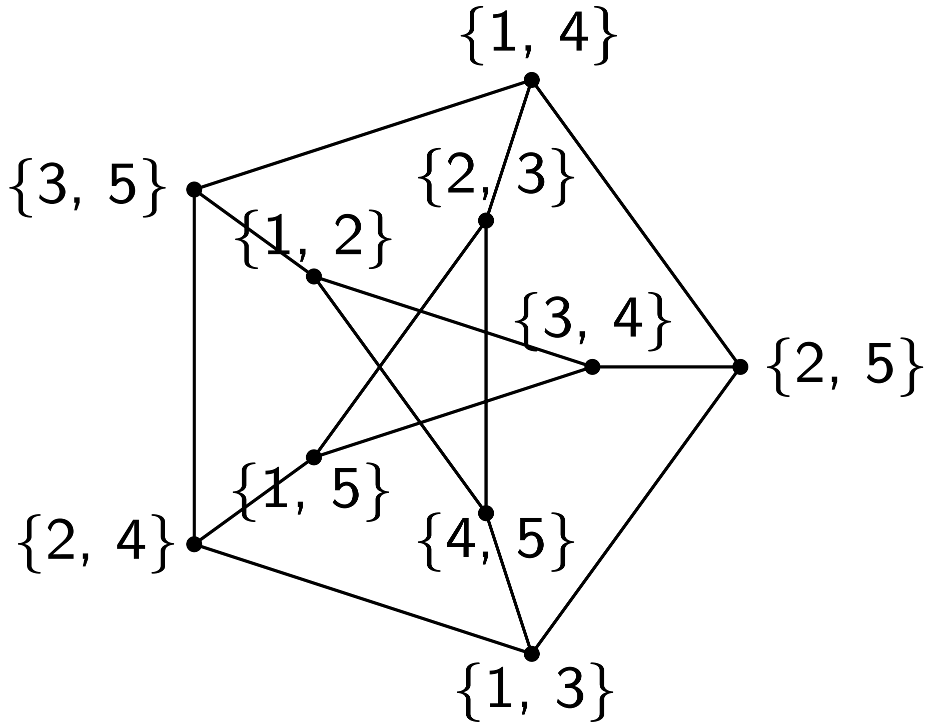

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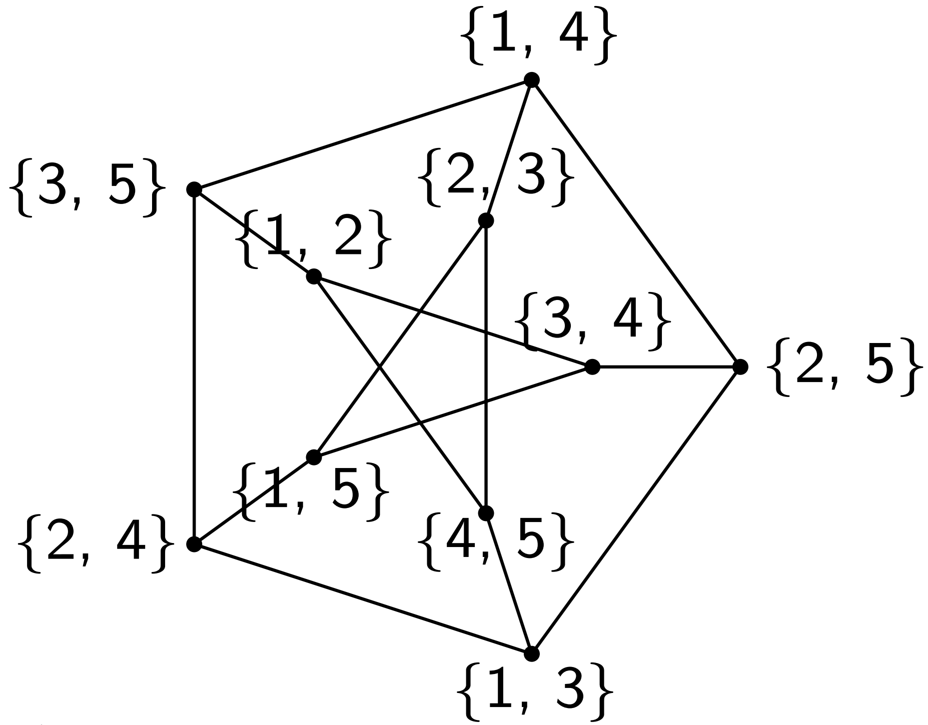

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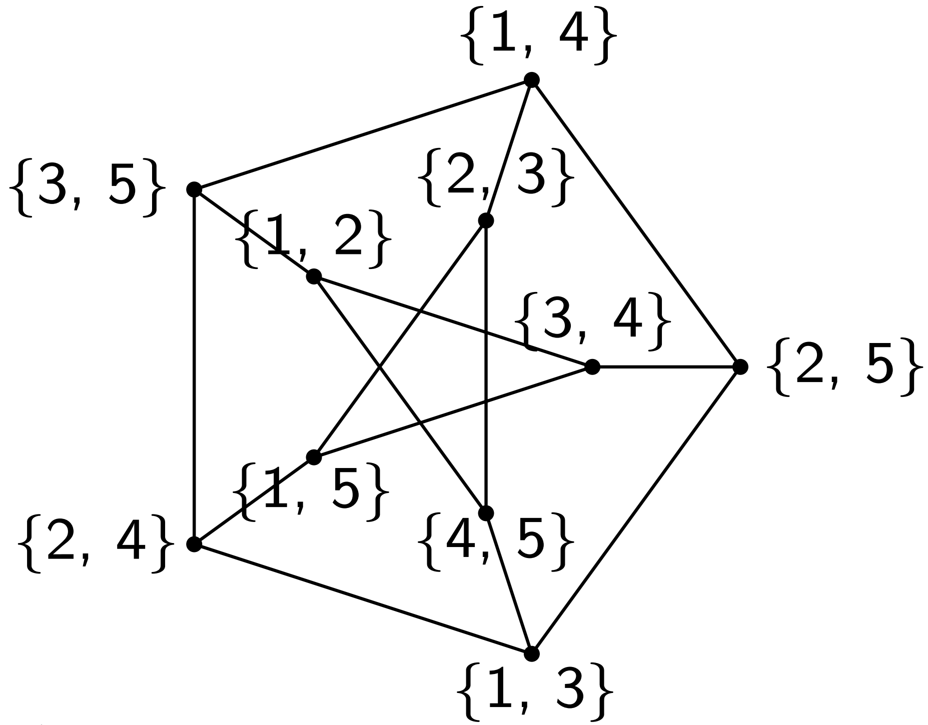

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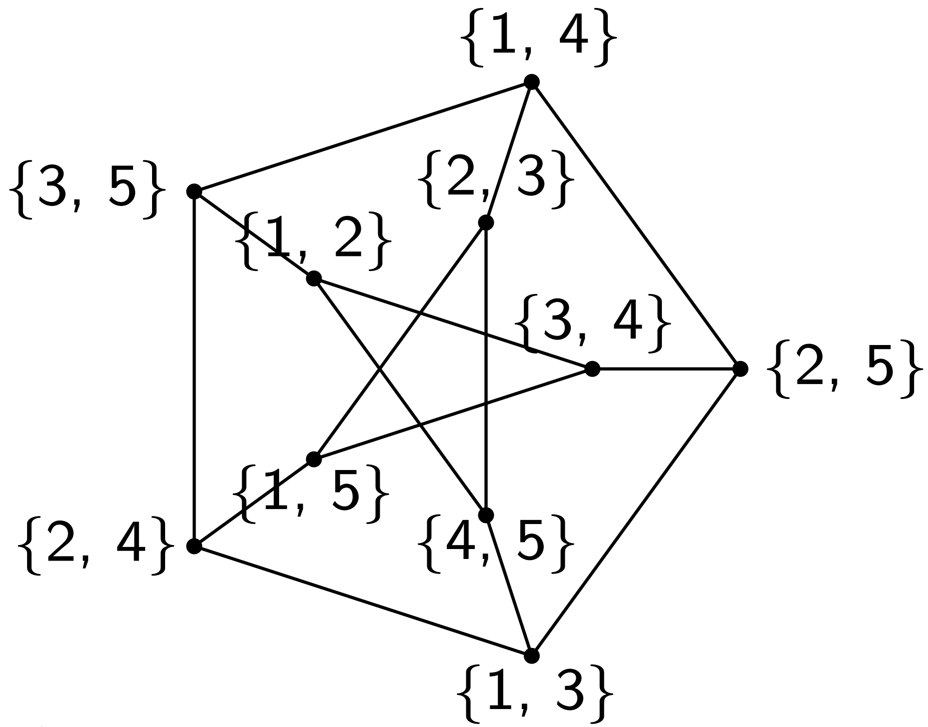

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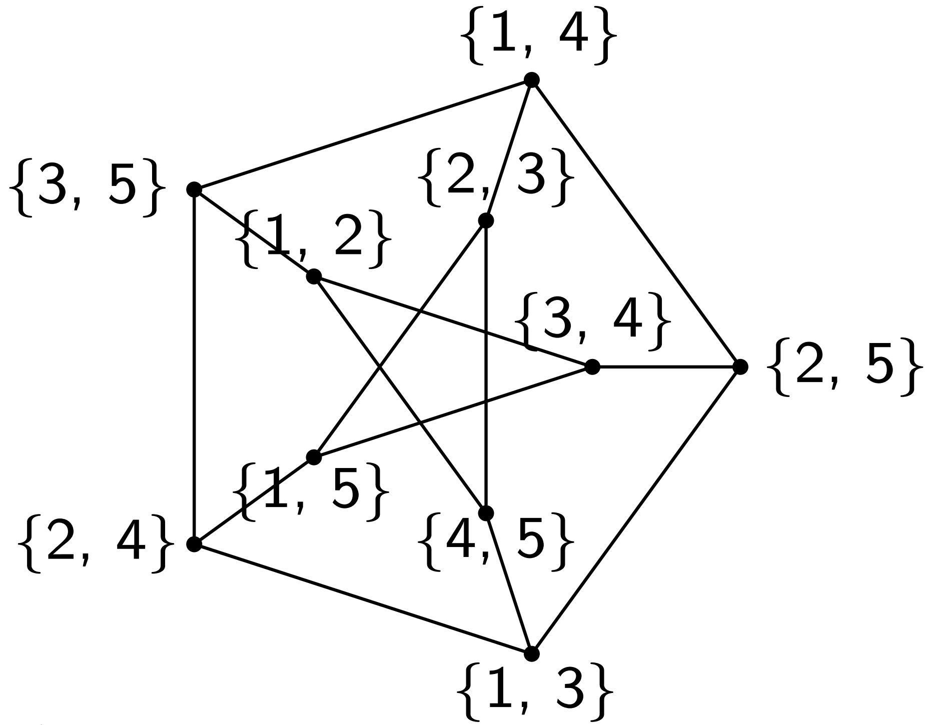

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The Rugby Footballers of Croam

MICHAEL MATHER

Department of Mathematics, University of Otago, Dunedin, New Zealand

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The vertices of the graph O_8 are indexed by the 7-subsets of a 15-set. Two vertices are adjacent if and only if their labeling sets are disjoint. This paper demonstrates a Hamiltonian circuit in O_8.

The following Hamiltonian circuit for O_8 was discovered by the methods of Meredith and Lloyd [1], with the help of a computer.

1234567:	9	1	10	2	11	7	12	3	2	4	13	9	14	10	15	11
	7	12	8	15	9	5	10	6	15	7	3	8	4	10	5	13
	6	14	7	2	10	3	12	4	13	5	1	6	2	10	3	12
	4	15	5	1	11	2	15	13	2	3	6	4	12	5	13	11
	3	12	5	2	10	6	14	13	11	7	8	11	4	10	1	9
	13	5	12	4	11	15	4	12	15	13	5	14	6	15	9	4
	12	5	13	8	15	11	10	9	4	13	7	15	8	3	14	6
	9	7	2	8	5	10	6	1	7	4	13	5	8	9	5	8
	11	14	15	7	10	13	9	12	4	11	14	6	12	5	8	15
	7	9	6	7	13	12	3	4	7	3	15	6	3	13	4	14
	6	4	11	7	4	2	12	3	13	4	5	13	4	12	1	11
	14	5	13	1	7	10	12	15	5	8	3	6	2	4	11	14
	10	13	9	12	1	11	15	9	14	3	13	2	12	1	8	15
	4	14	3	10	2	8	1	5	11	4	10	3	9	12	6	2
	5	11	4	10	3	7	14	6	12	5	11	4	8	13	7	1
	6	12	5	9	2	8	12	7	10	6	9	12	8	11	7	10
	1	9	13	8	12	7	6	12	7	13	8	1	9	3	10	5
	11	6	12	7	15	8	4	9	1	10	5	2	6	11	9	12
	10	4	2	14	12	2	4	6	7	10	14	15	11	5	2	11
	8	9	10	12	13	14	15	1	14	15	6	8	12	7	9	14
	5	10	1	6	8	5	14	2	7	14	6	9	3	8	11	7
	10	13	9	12	15	11	14	4	13	1	7	15	5	14	2	10
	1	7	15	1	12	2	10	13	2	6	8	2	7	10	2	8
	14	3	13	15	4	12	10	4	15	10	11	15	10	14	4	11
	1	7	15	5	8	10	12	2	6	4	5	1	4	9	3	8
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582b/20/1-5

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The Rugby Footballers of Croam

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Department of Mathematics, University of Otago, Dunedin, New Zealand

Communicated by W. T. Tutte

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	7	12	8	15	9	5	10	6	15	7	3	8	4	10	5	13
	6	14	7	2	10	3	12	4	13	5	1	6	2	10	3	12
	4	15	5	1	11	2	15	13	2	3	6	4	12	5	13	11
	3	12	5	2	10	6	14	13	11	7	8	11	4	10	1	9
	13	5	12	4	11	15	4	12	15	13	5	14	6	15	9	4
	12	5	13	8	15	11	10	9	4	13	7	15	8	3	14	6
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	11	14	15	7	10	13	9	12	4	11	14	6	12	5	8	15
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	13	5	12	4	11	15	4	12	15	13	5	14	6	15	9	4
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	4	14	3	10	2	8	1	5	11	4	10	3	9	12	6	2
	5	11	4	10	3	7	14	6	12	5	11	4	8	13	7	1
	6	12	5	9	2	8	12	7	10	6	9	12	8	11	7	10
	1	9	13	8	12	7	6	12	7	13	8	1	9	3	10	5
	11	6	12	7	15	8	4	9	1	10	5	2	6	11	9	12
	10	4	2	14	12	2	4	6	7	10	14	15	11	5	2	11
	8	9	10	12	13	14	15	1	14	15	6	8	12	7	9	14
	5	10	1	6	8	5	14	2	7	14	6	9	3	8	11	7
	10	13	9	12	15	11	14	4	13	1	7	15	5	14	2	10
	1	7	15	1	12	2	10	13	2	6	8	2	7	10	2	8
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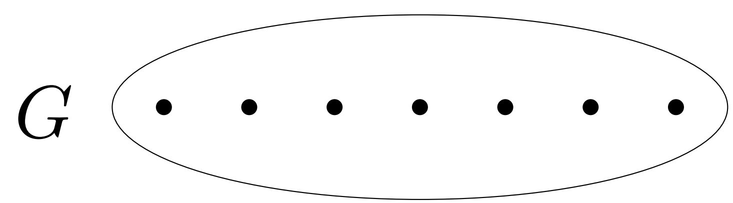

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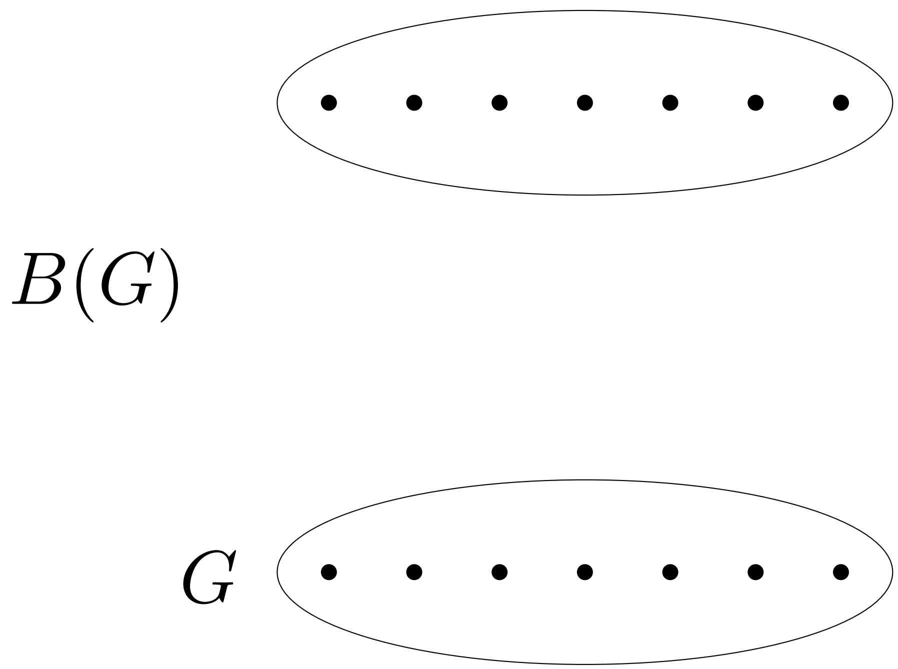

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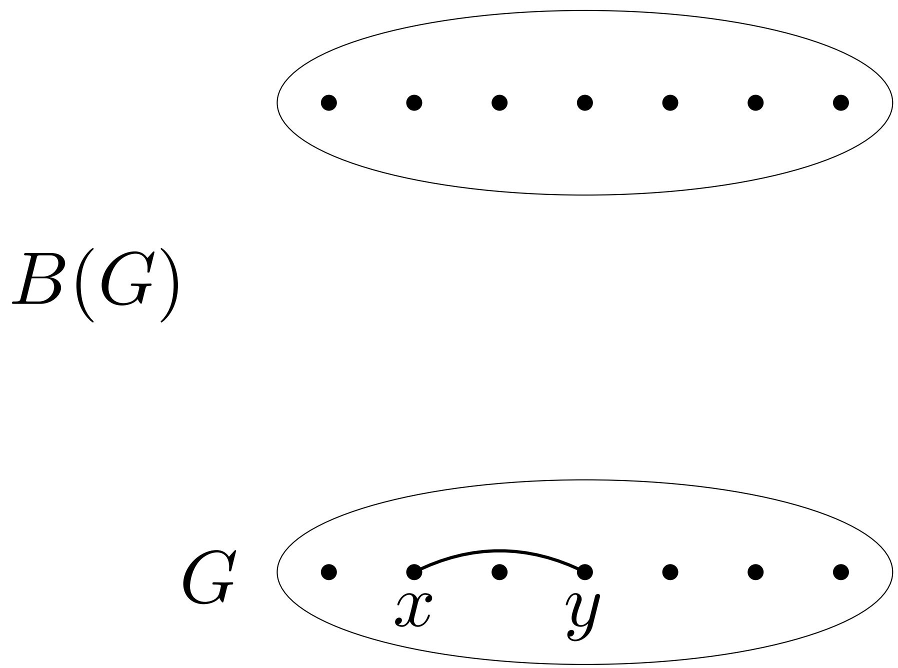

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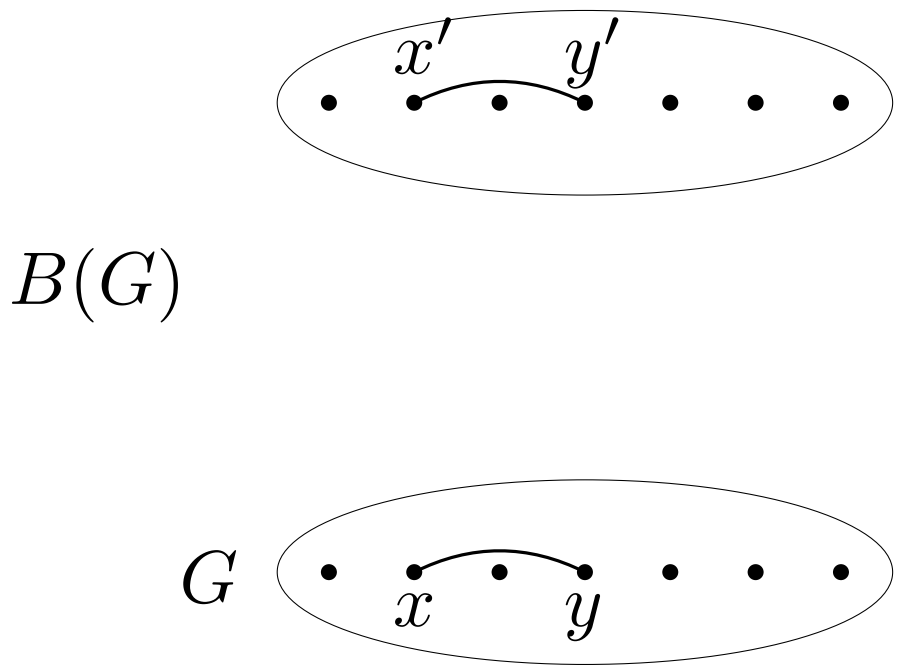

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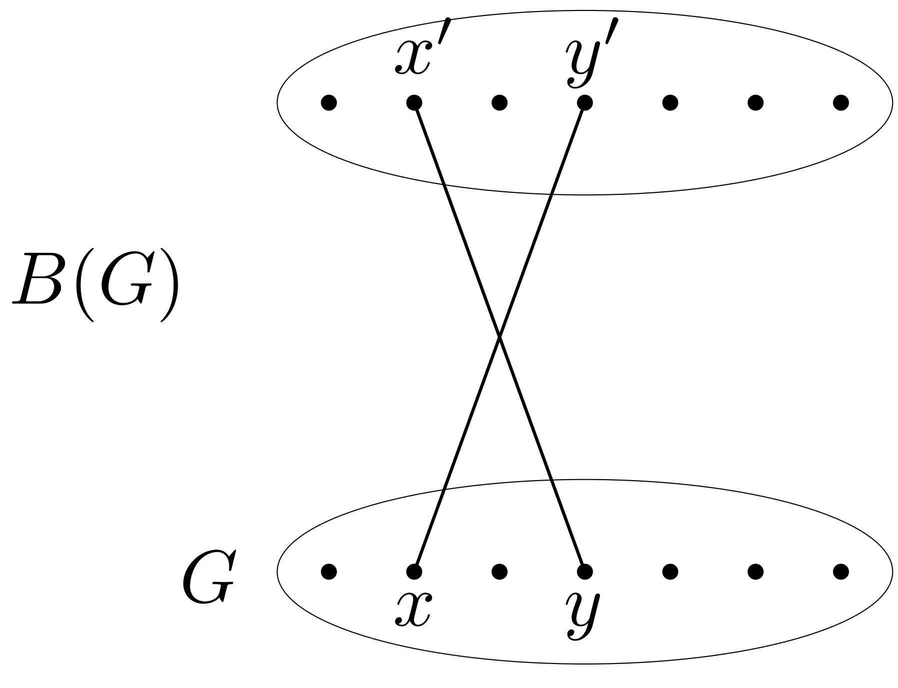

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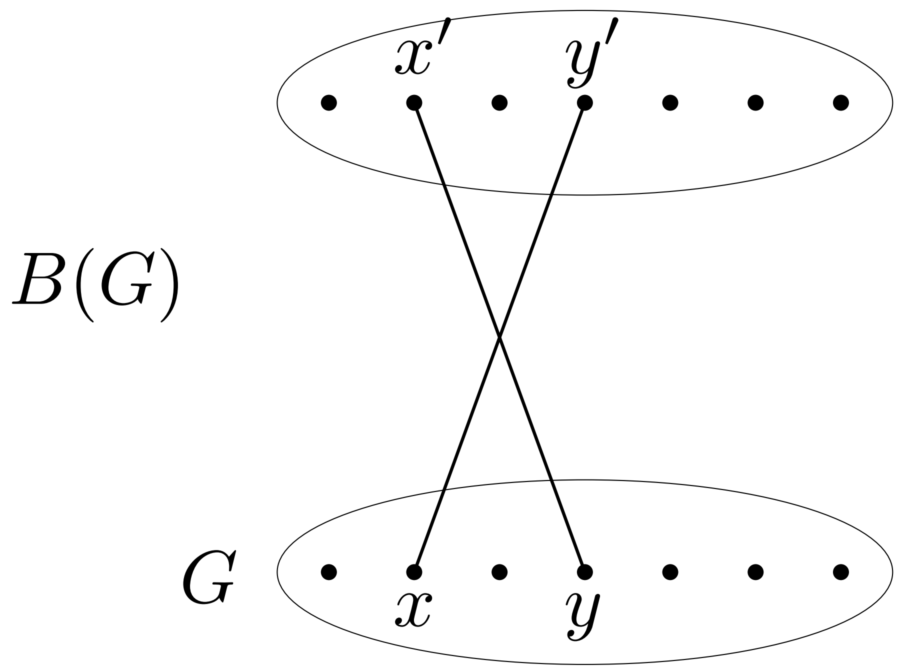

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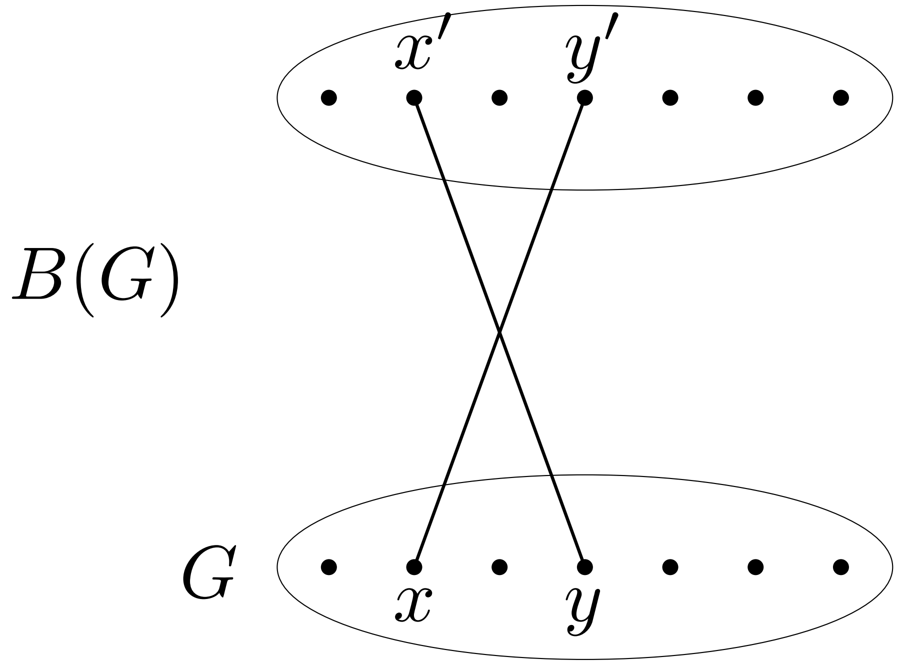

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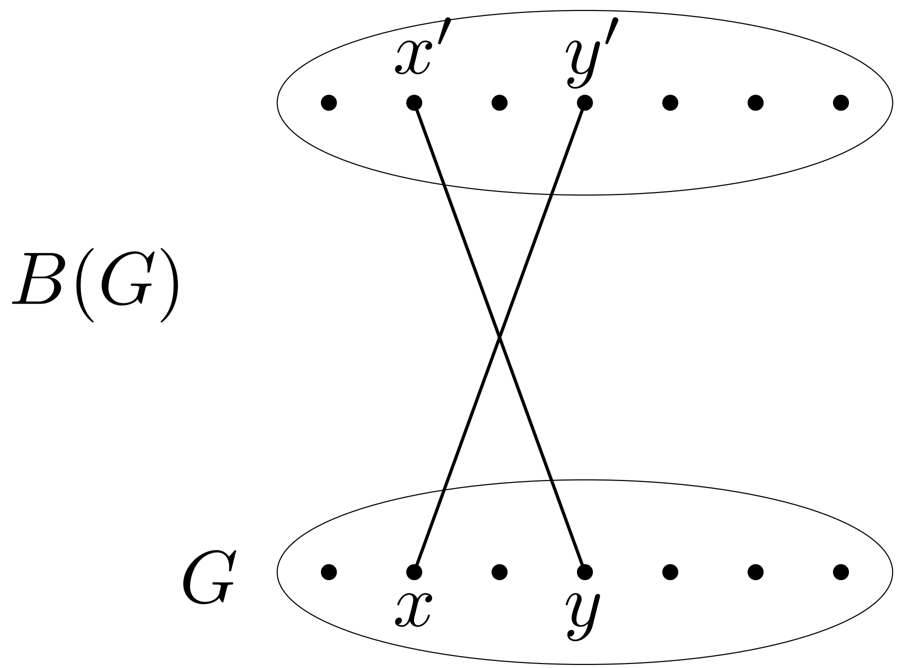
- we thus obtain a new proof for Hamiltonicity of $H_{n,k}$



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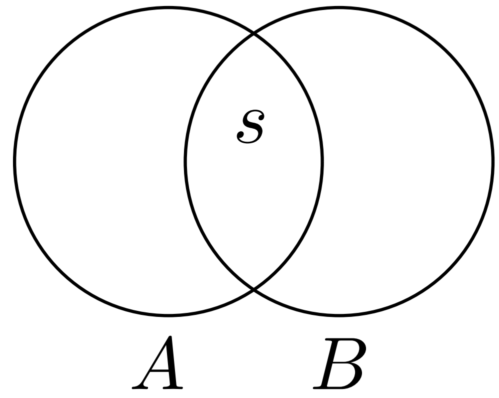

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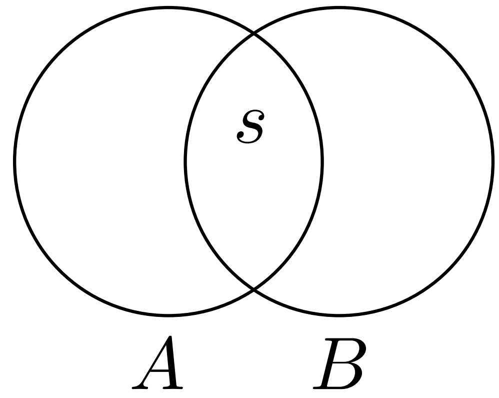

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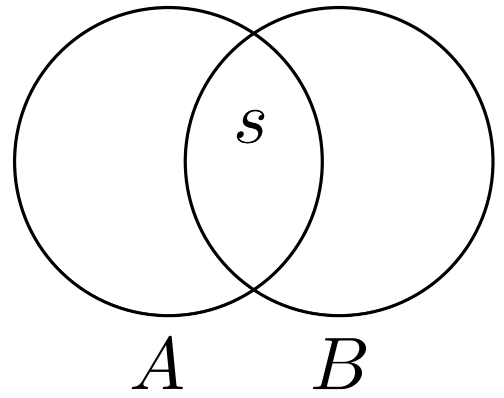

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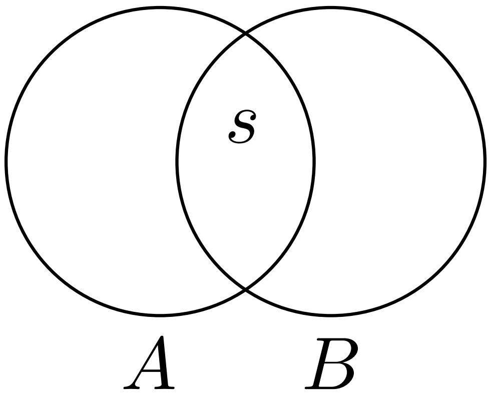

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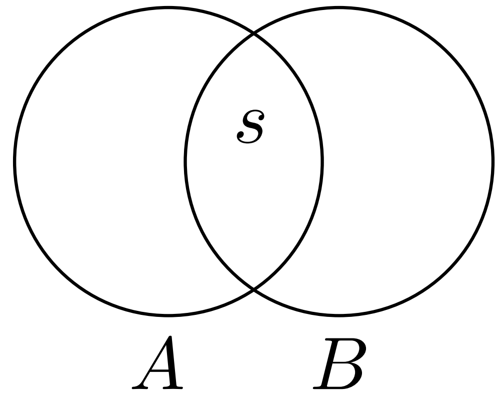

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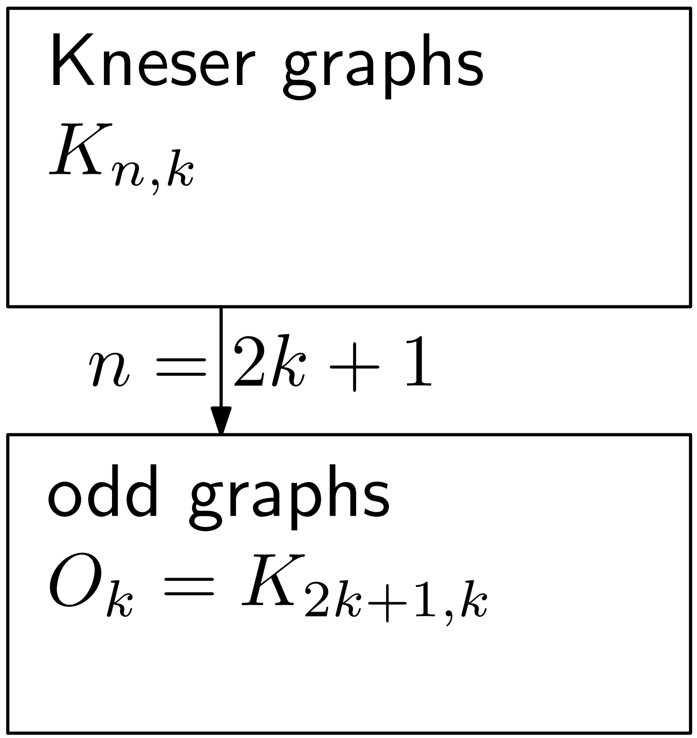

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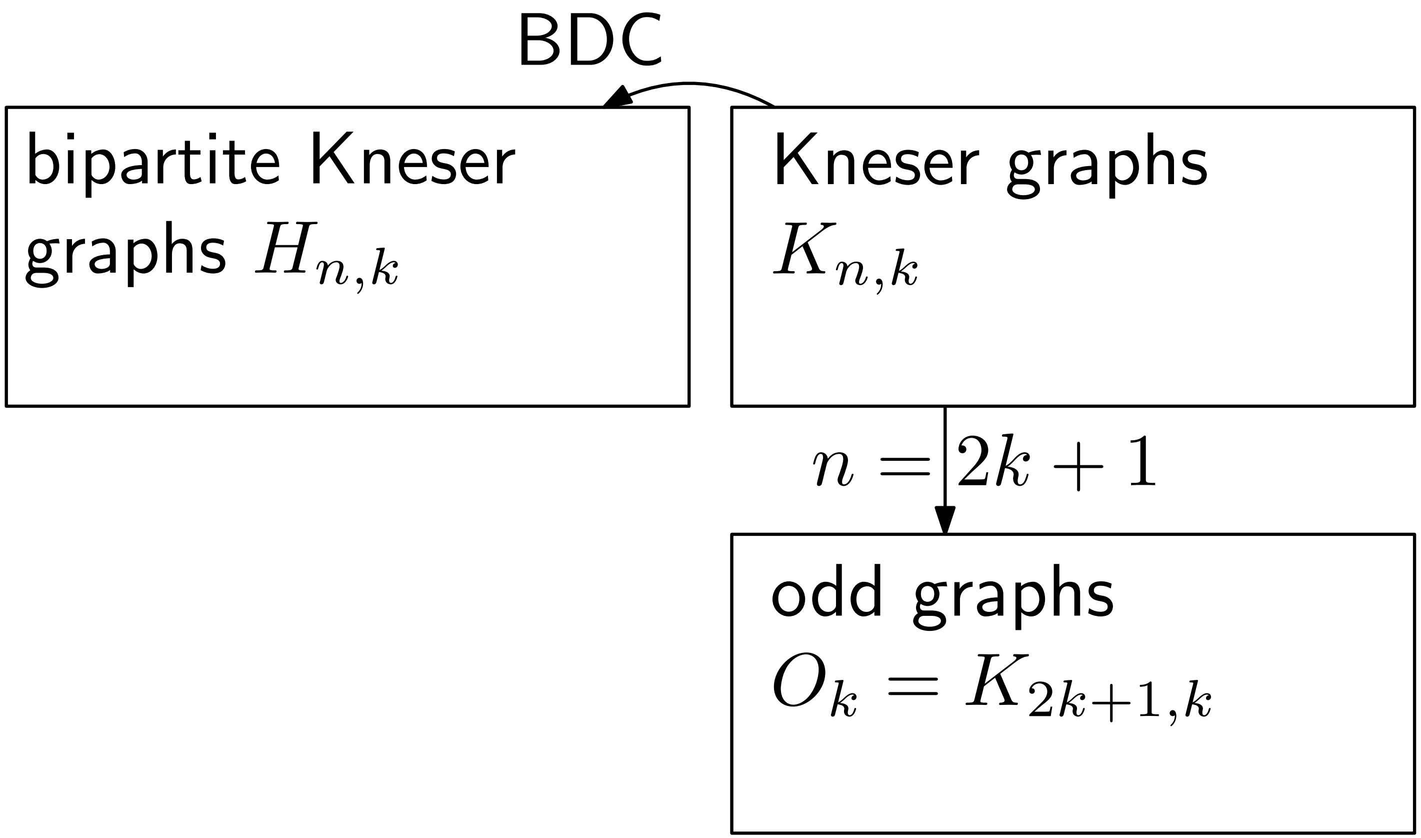

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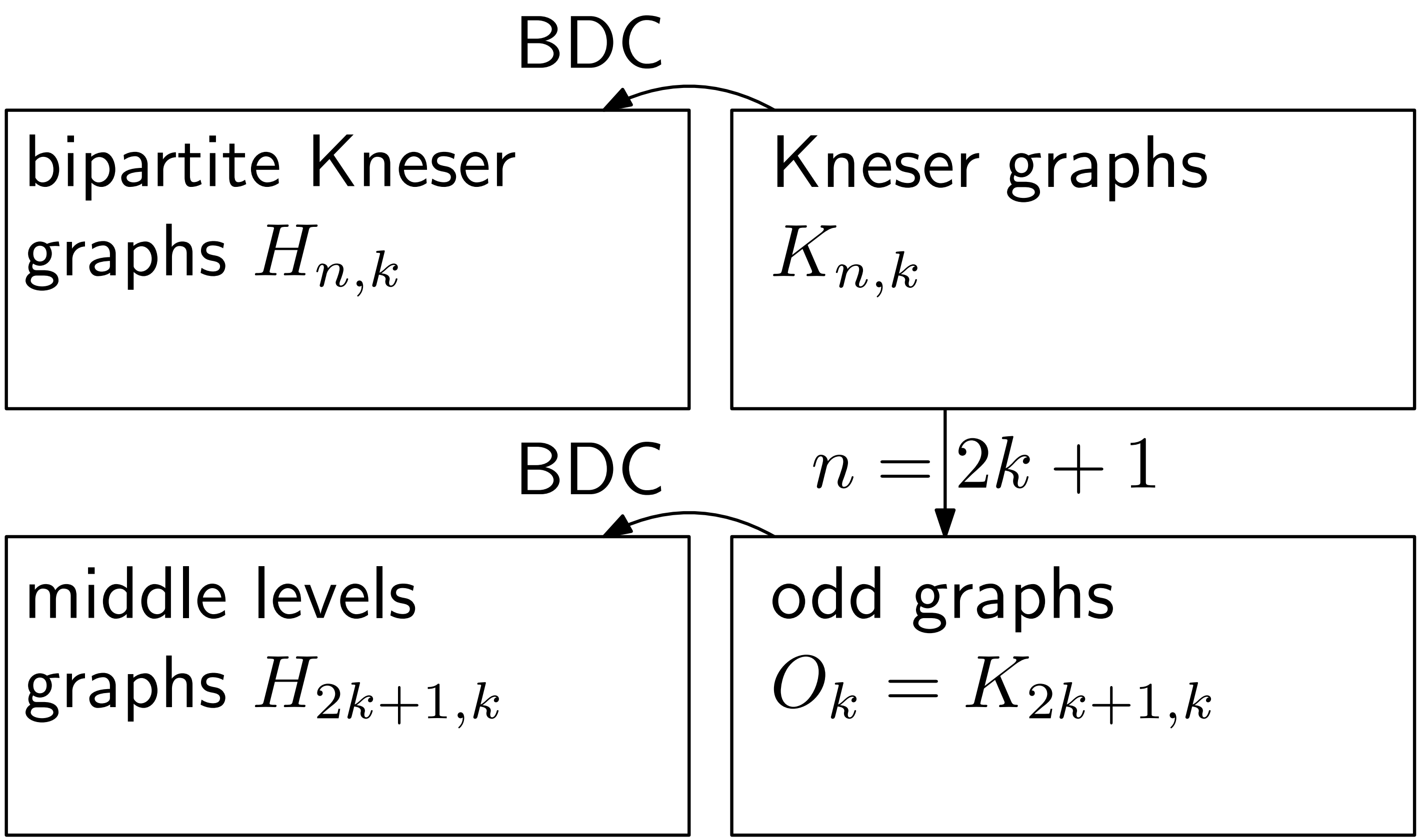

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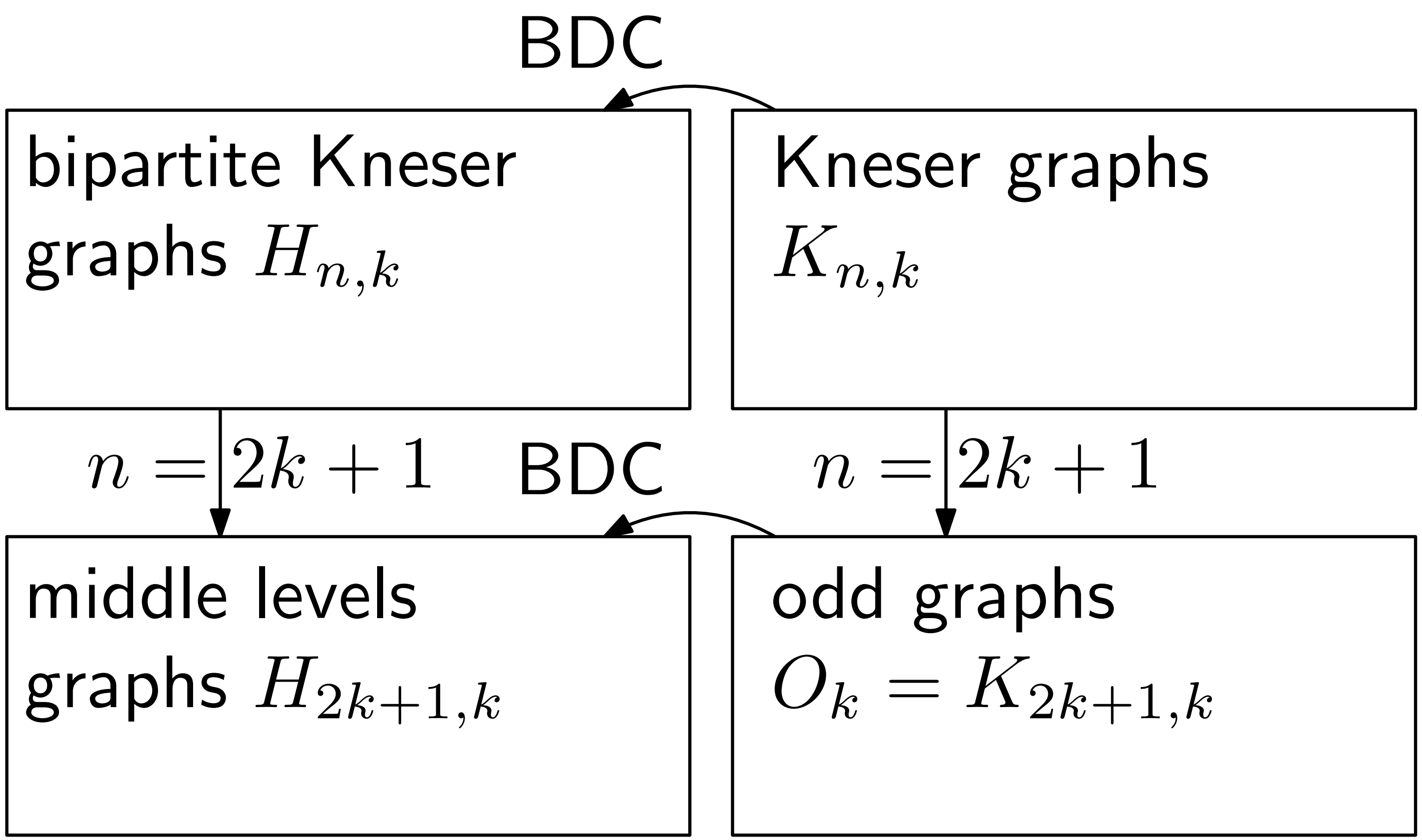

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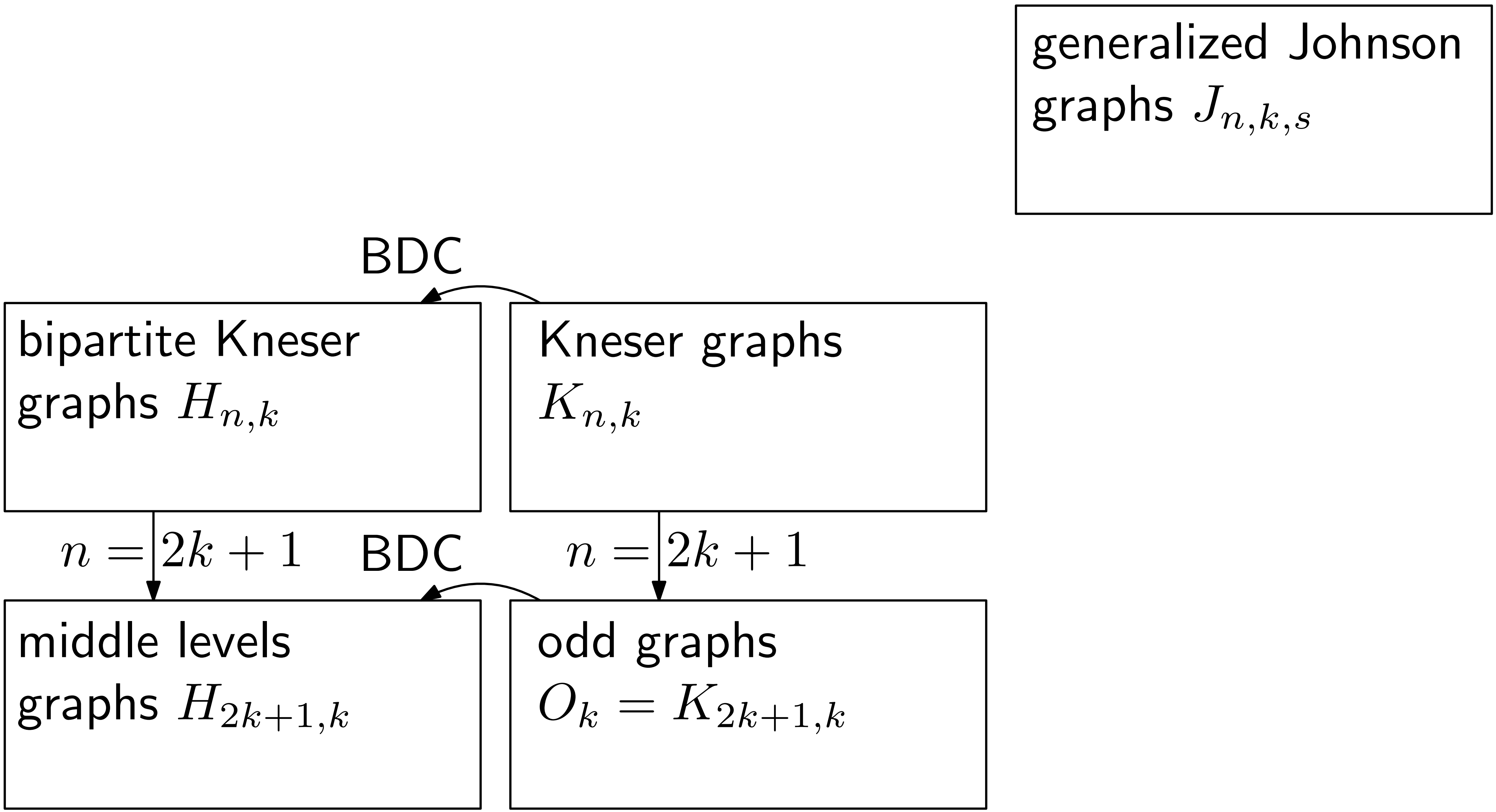

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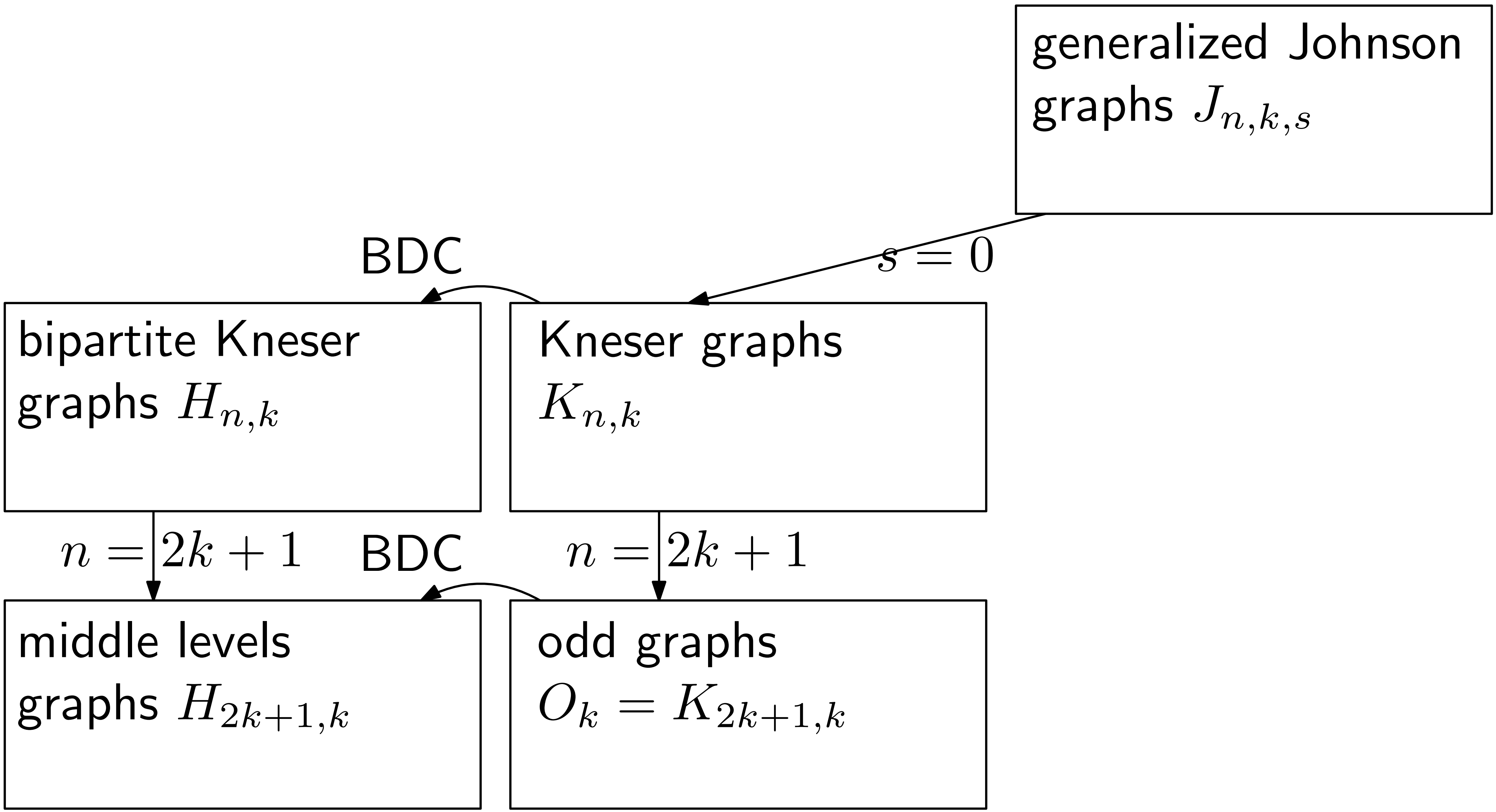

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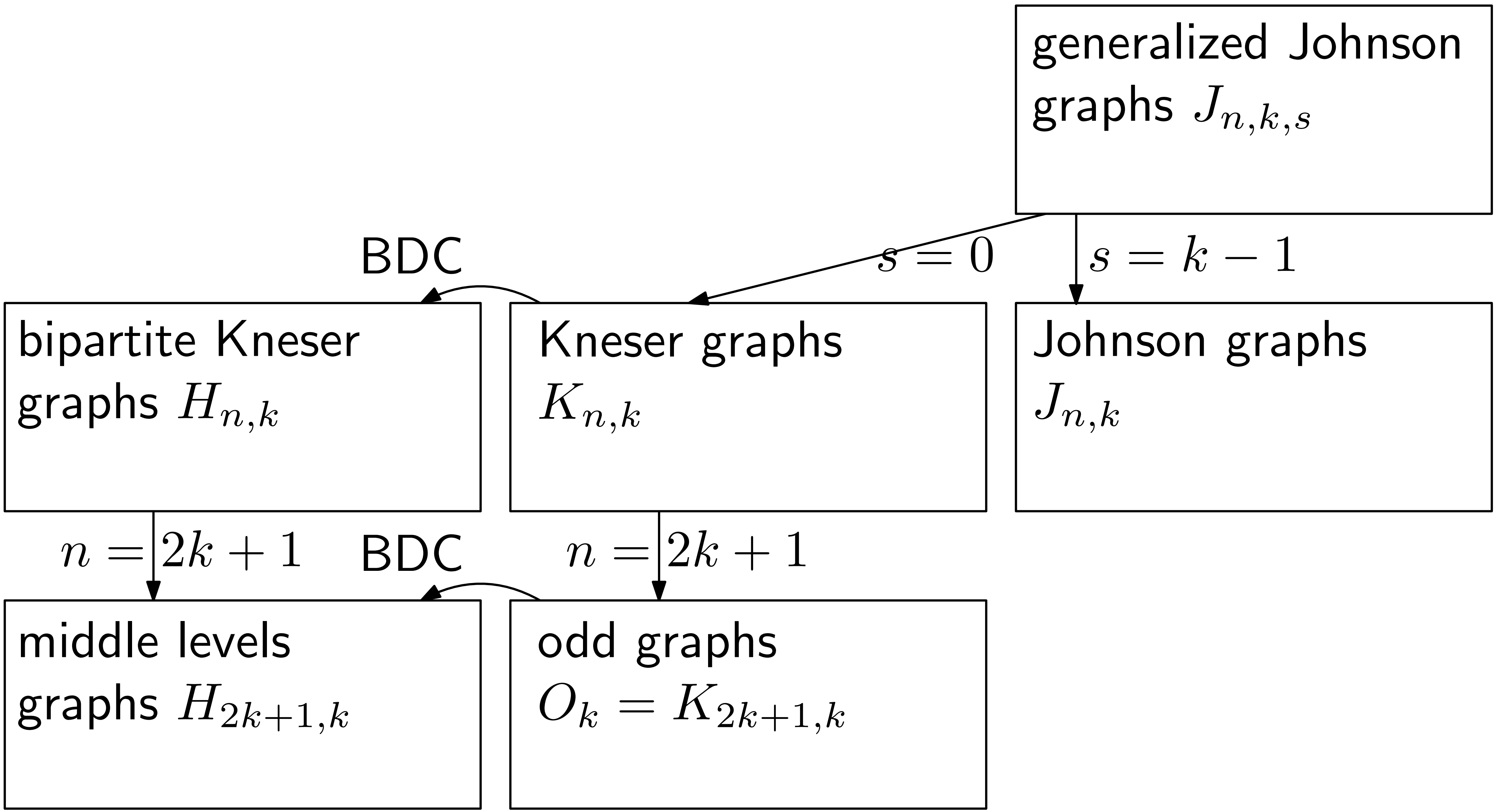

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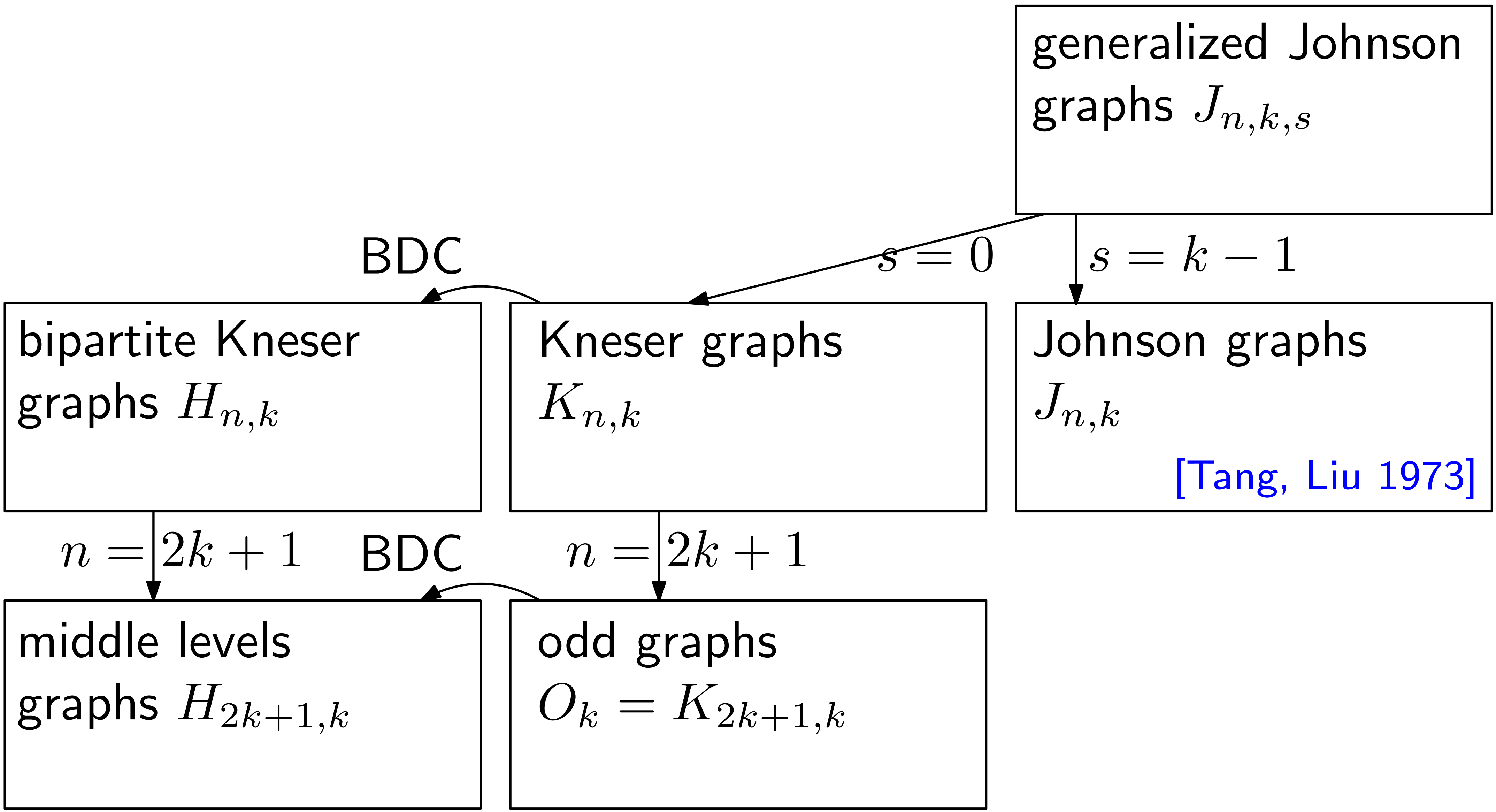

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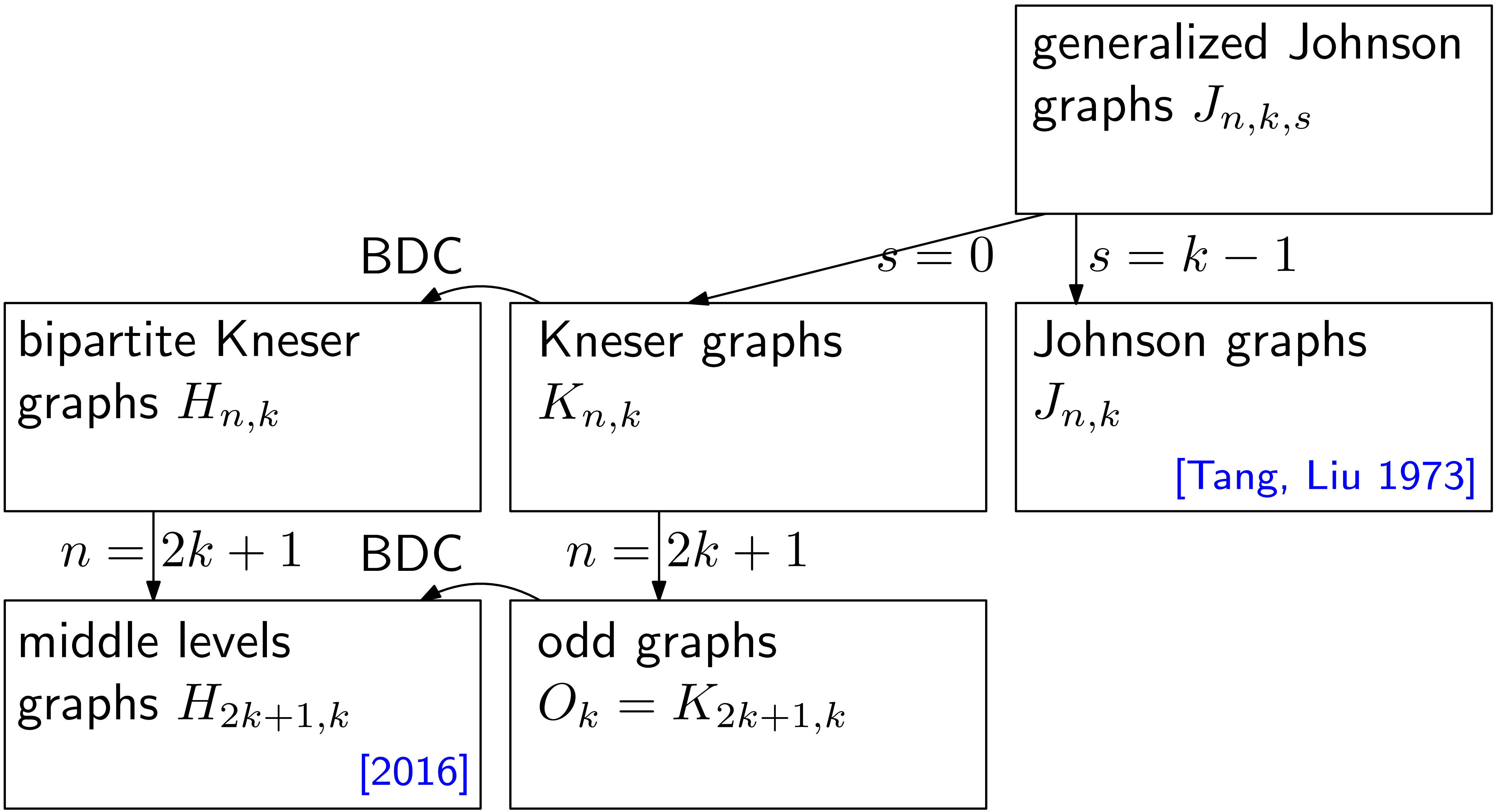

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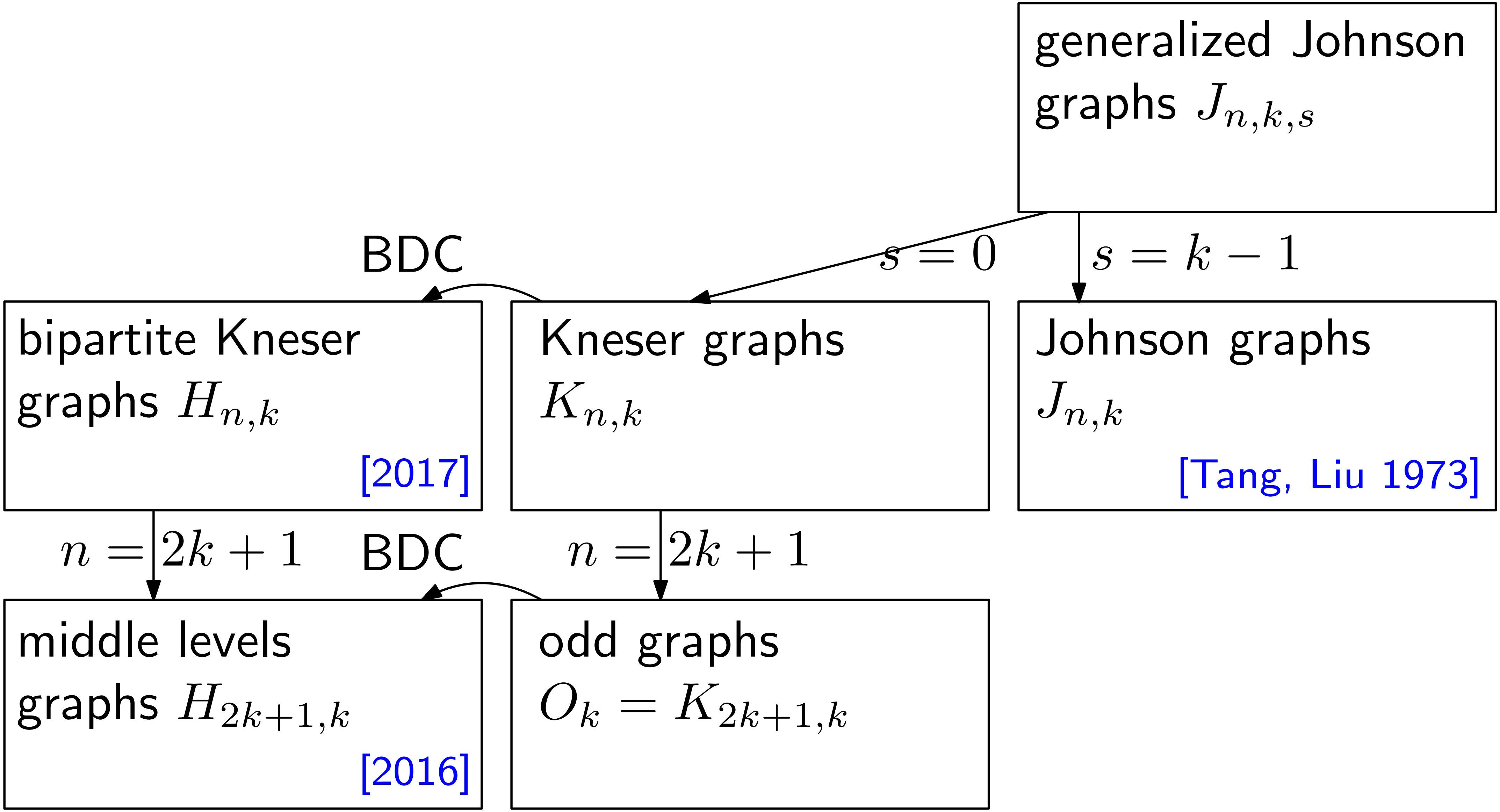

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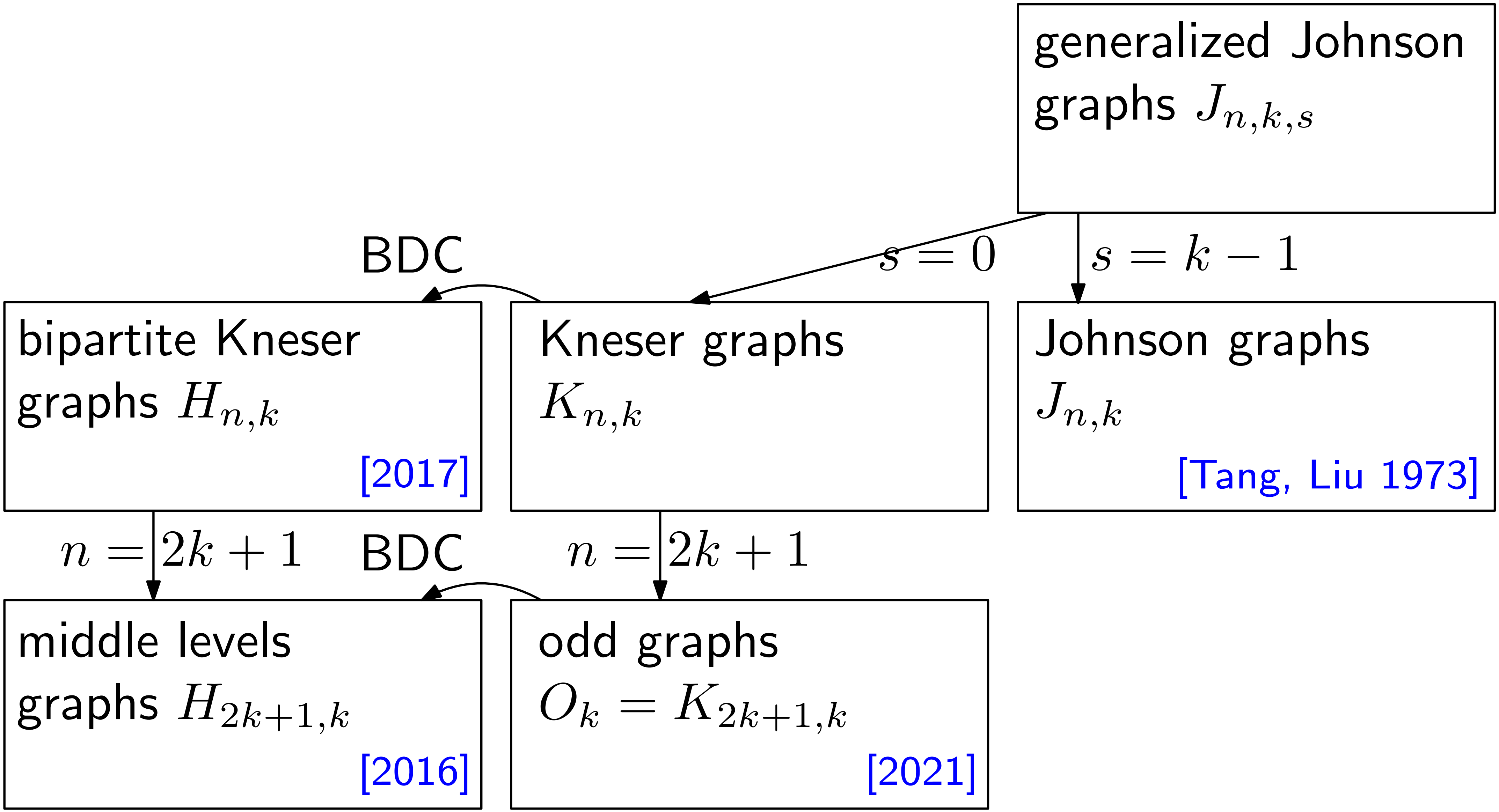

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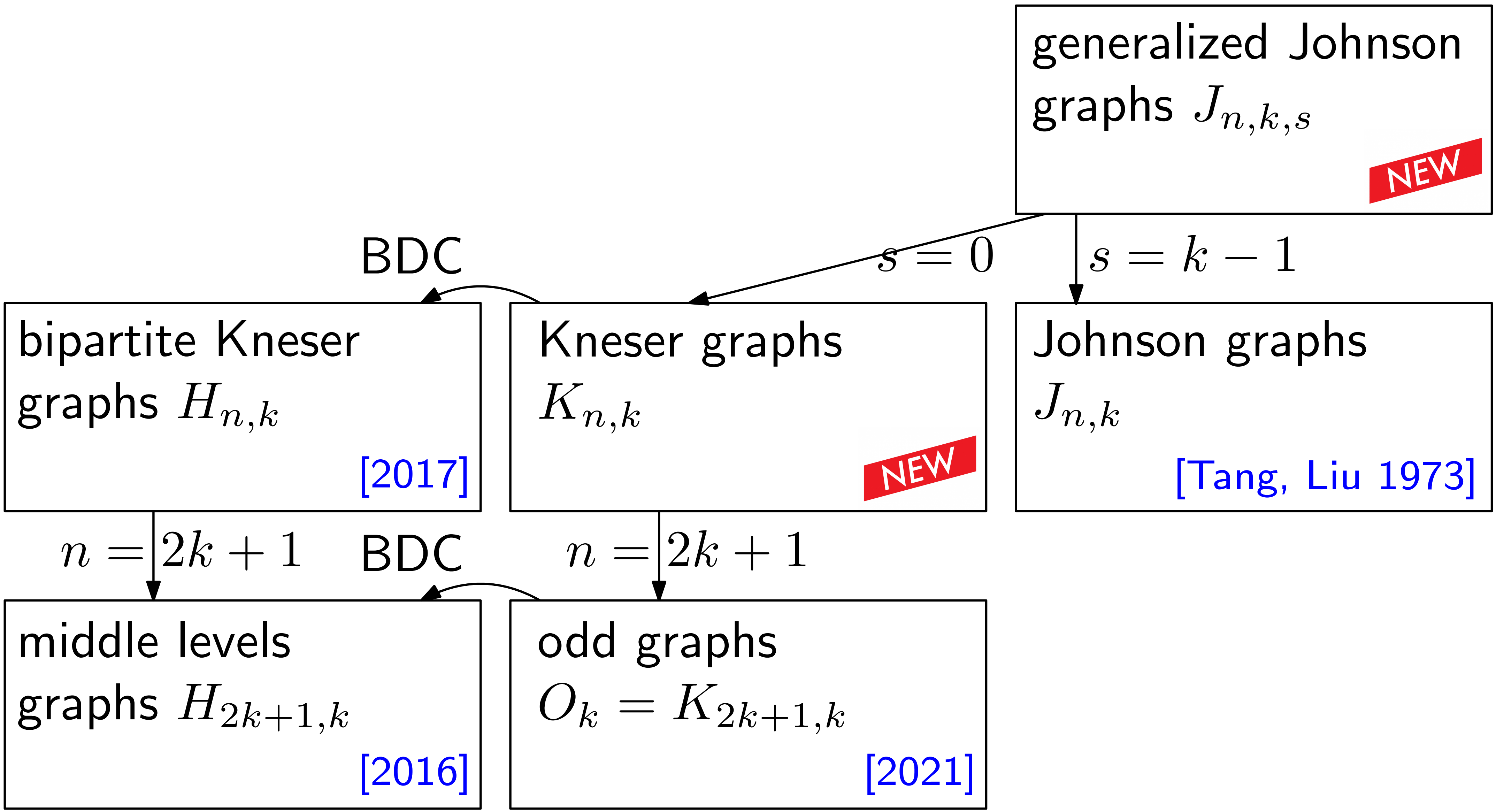

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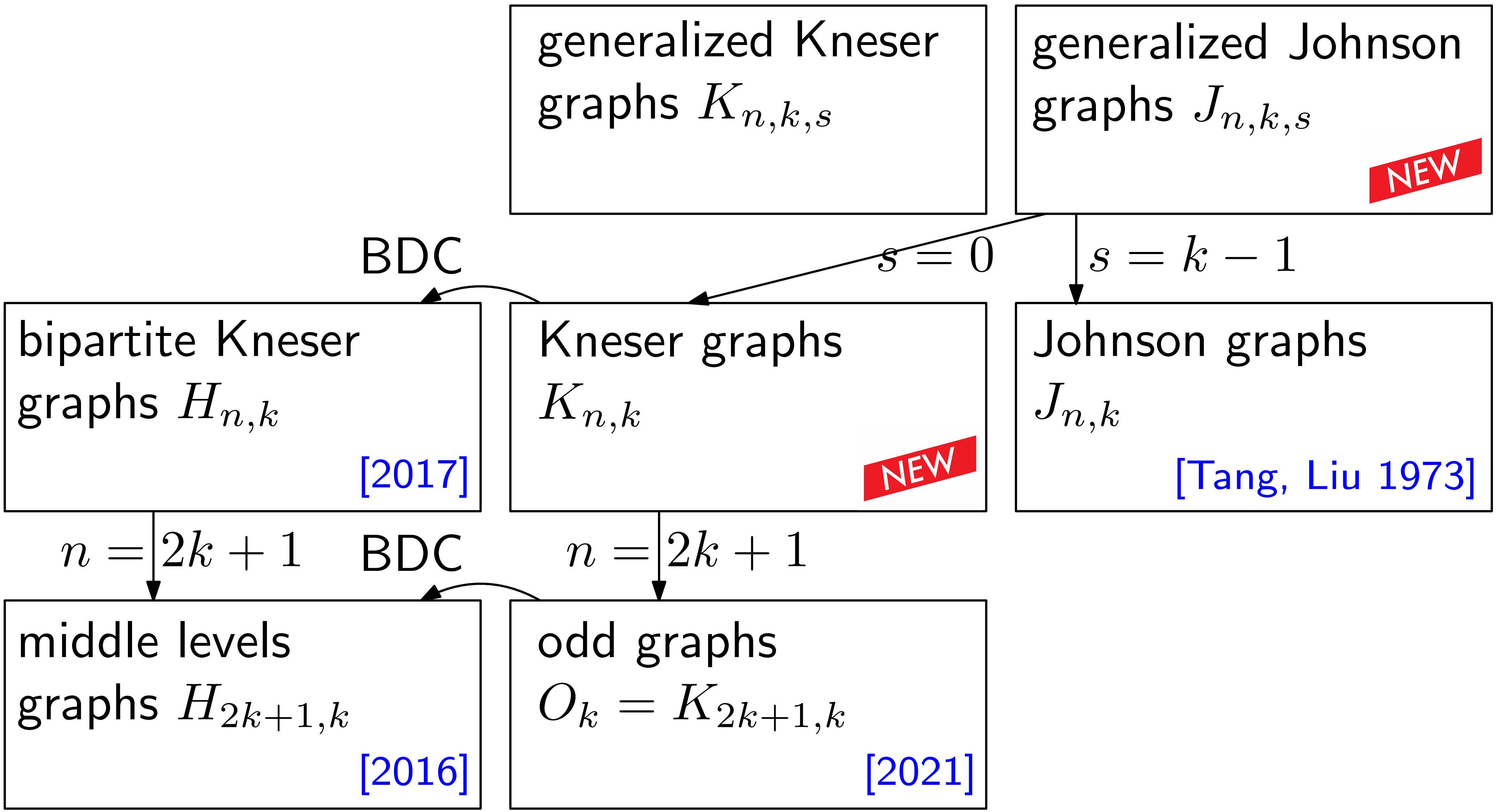

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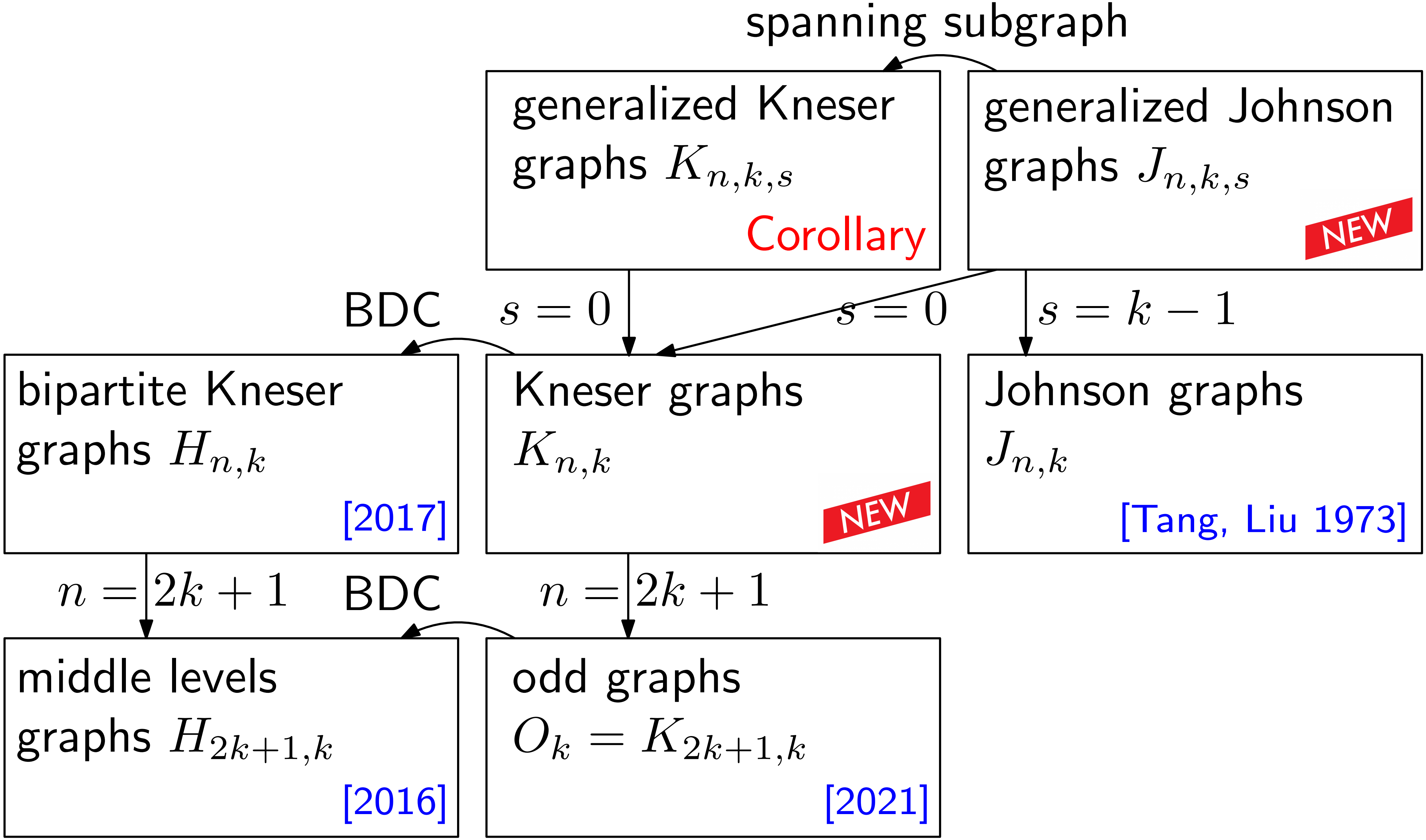

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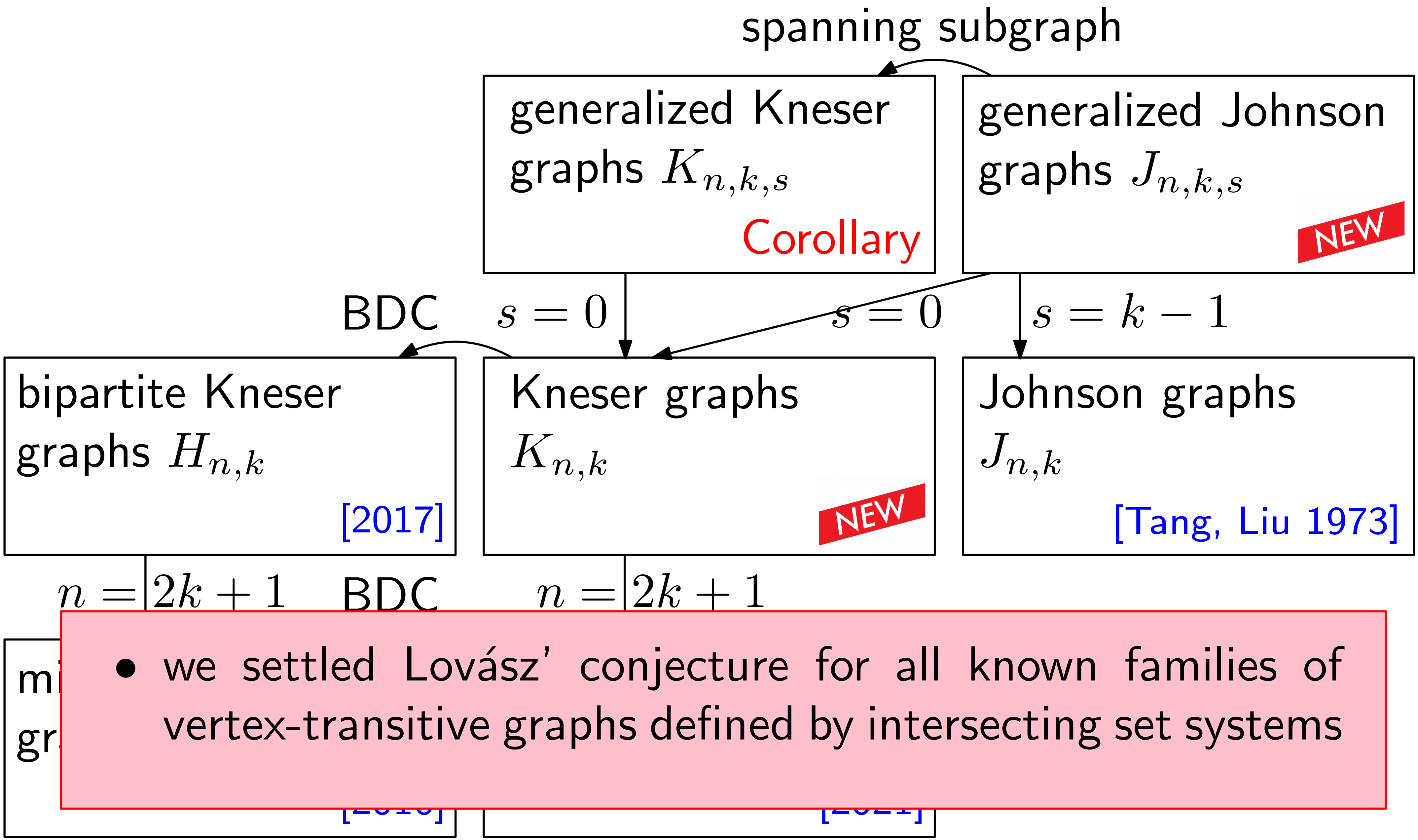

- we settled Lovász' conjecture for all known families of vertex-transitive graphs defined by intersecting set systems

Proof outline for $K_{n,k}$

- two sparsest cases $n = 2k + 1$ and $n = 2k + 2$ settled by

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  - reminiscent of the gliders in Conway's game of Life
  - main technical innovation

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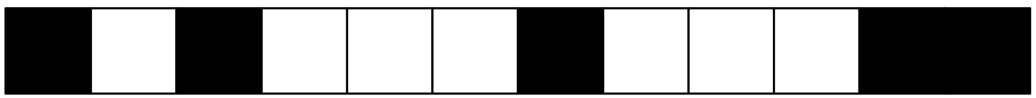
1 0 1 0 0 0 1 0 0 0 1 1



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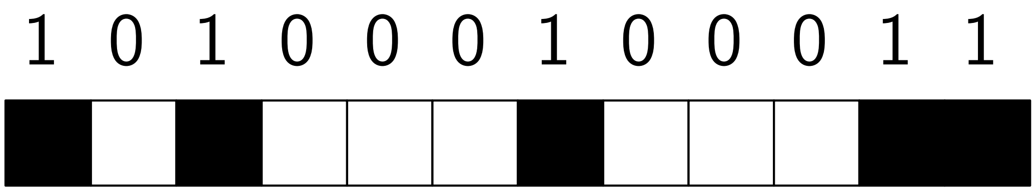

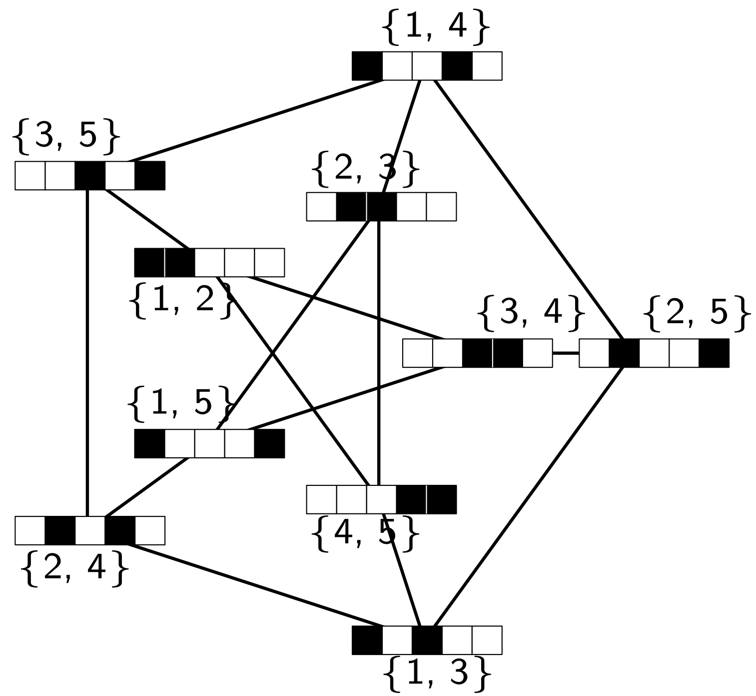

Cycle factor

- parenthesis matching with 1=[and 0=] (cyclically)

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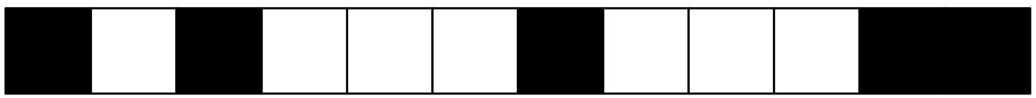

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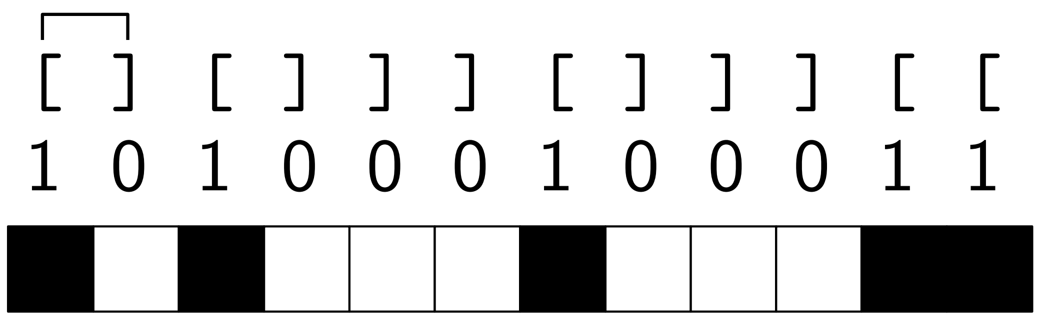

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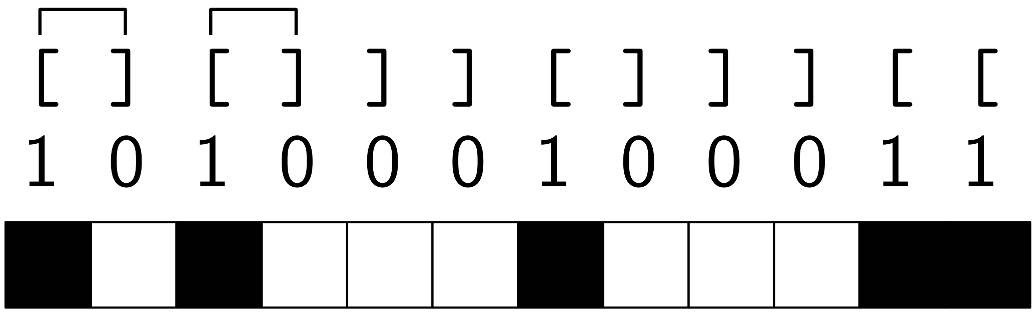

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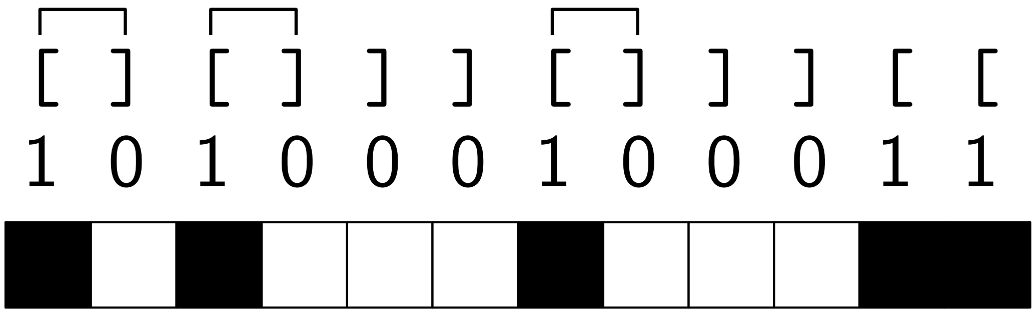

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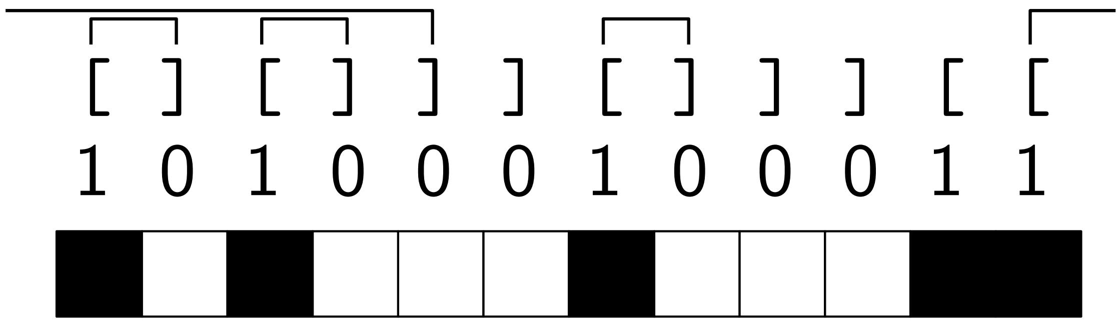

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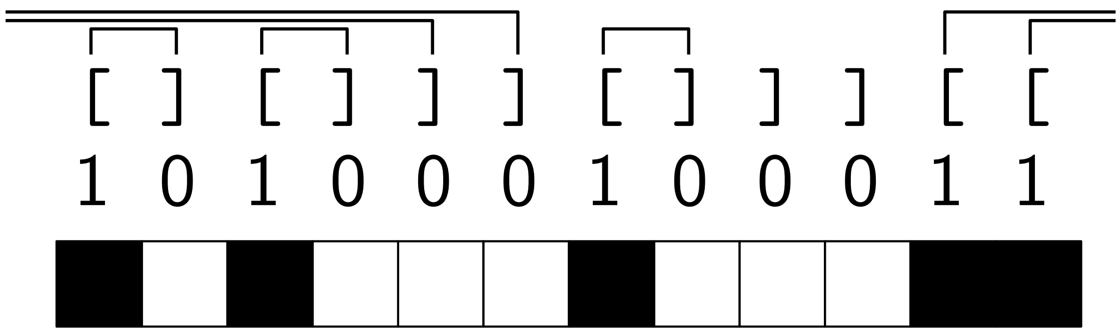

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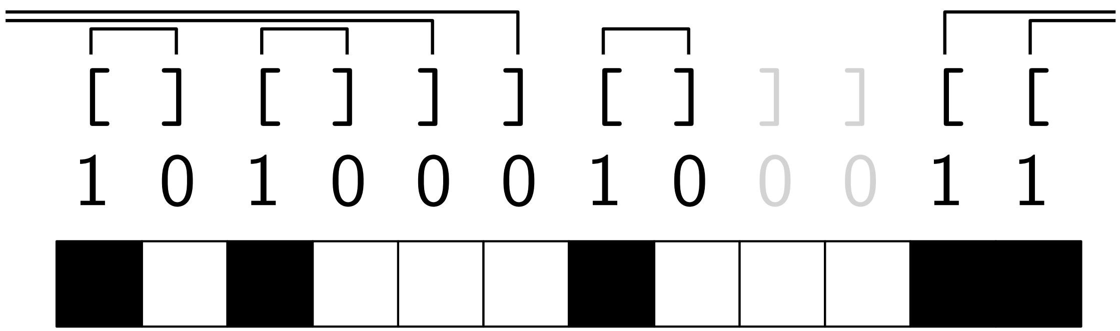

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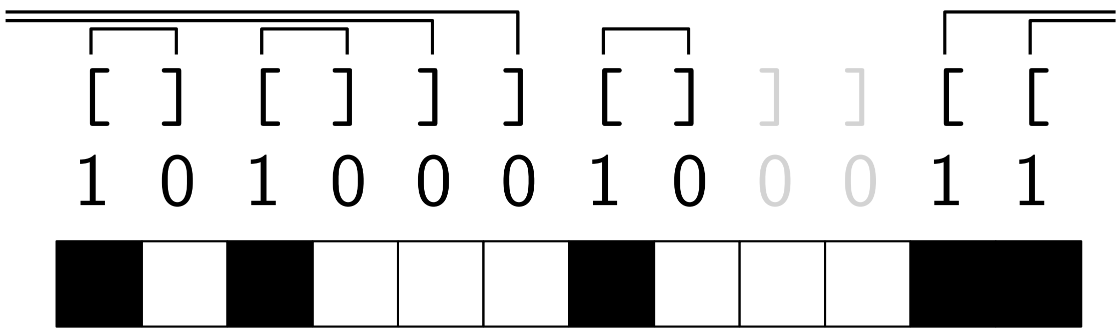

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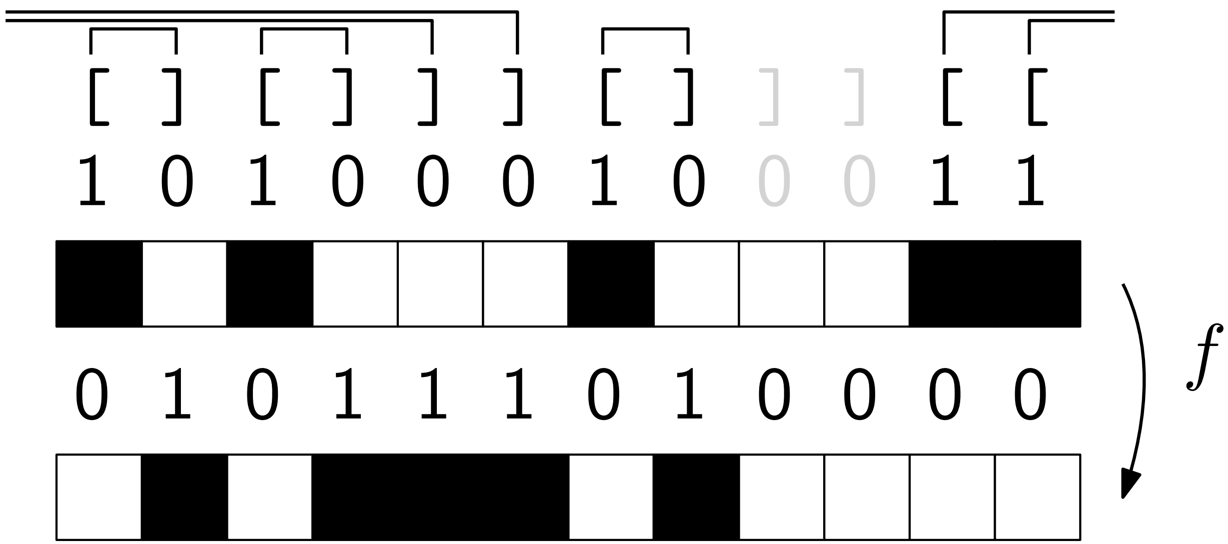

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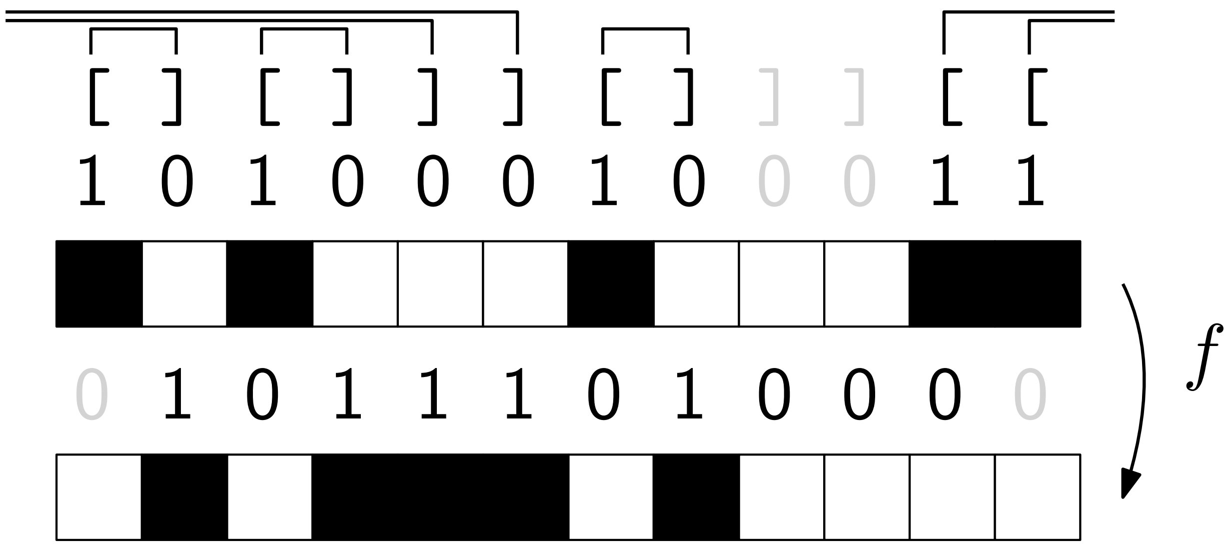

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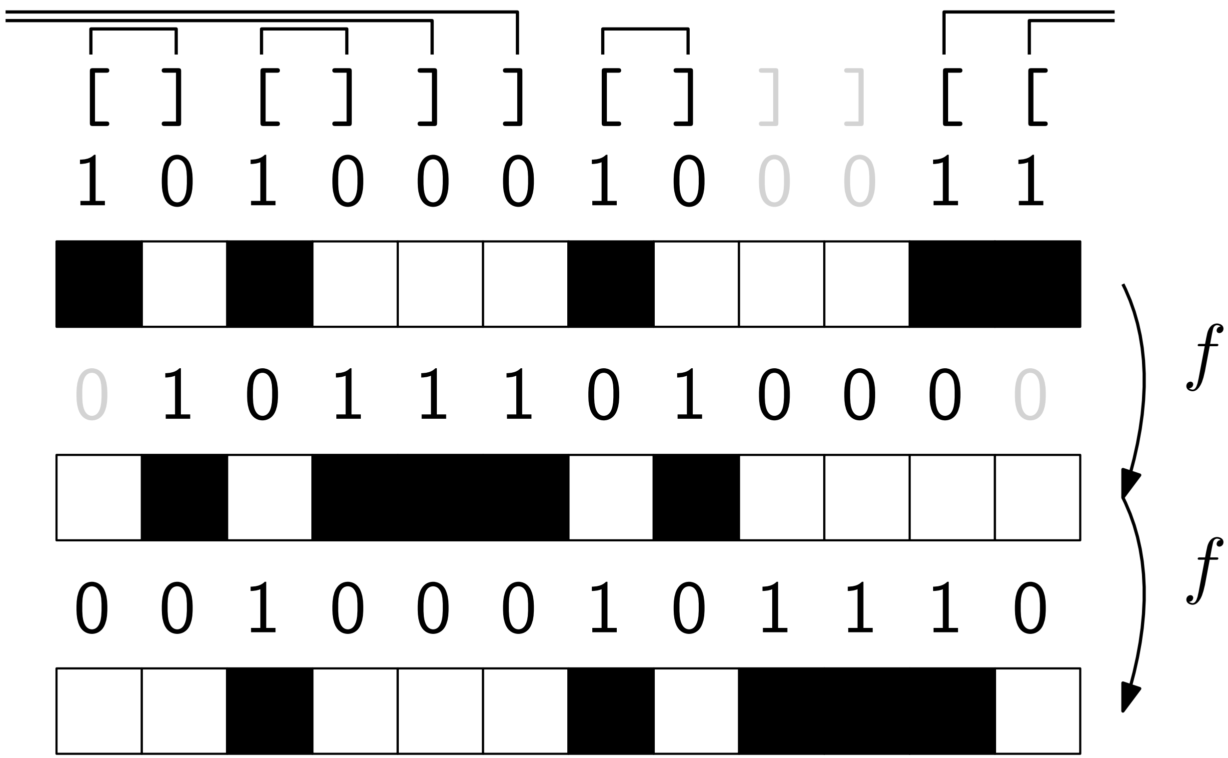

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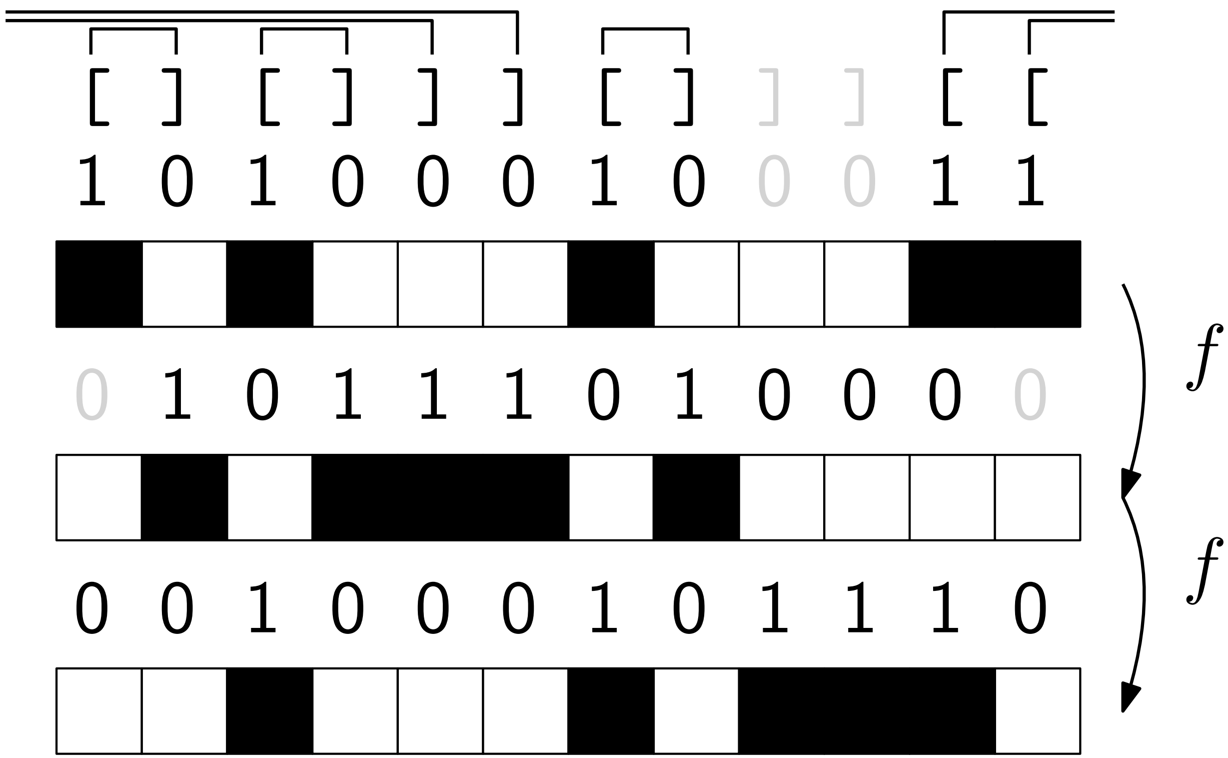
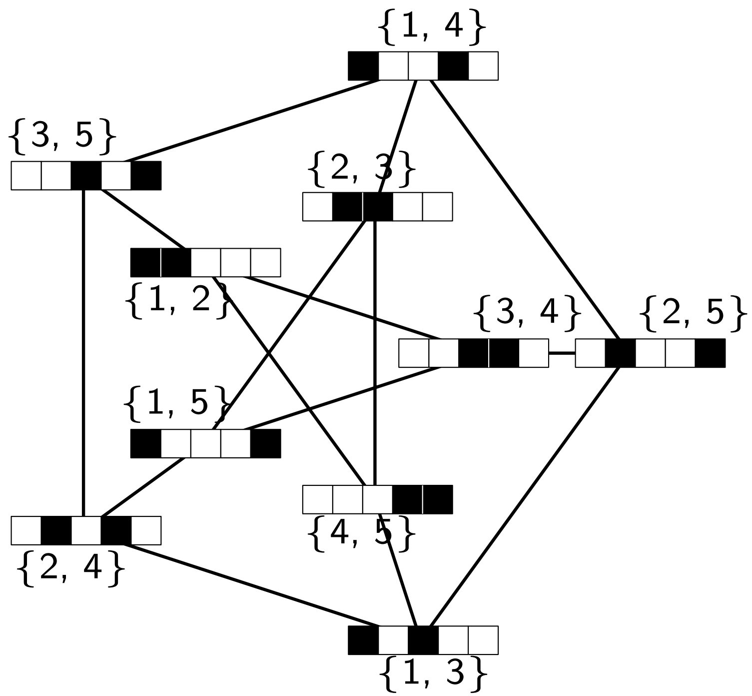


- f is invertible $\rightarrow$ partition of $K_{n,k}$ into disjoint cycles

Cycle factor

- **Example:** $K_{5,2}$



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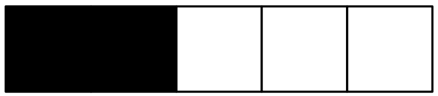

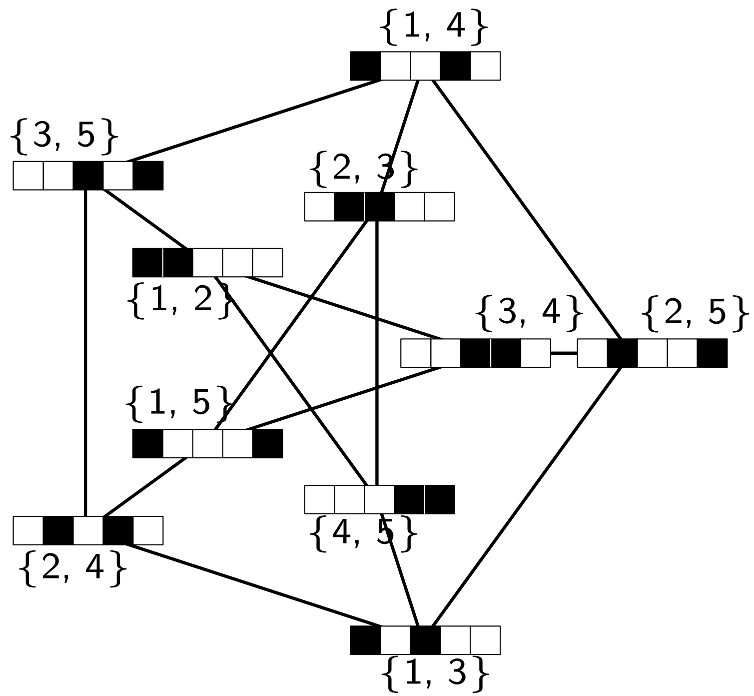

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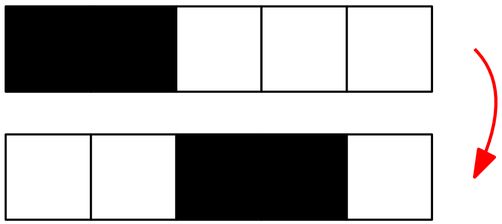

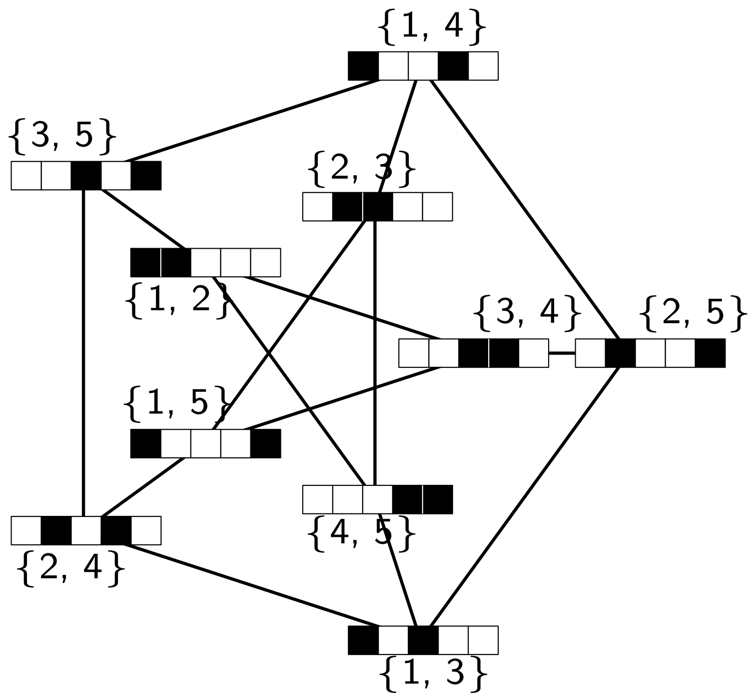

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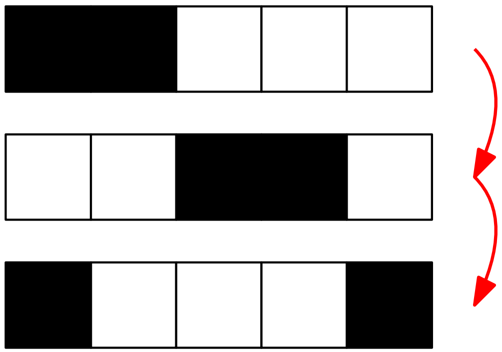

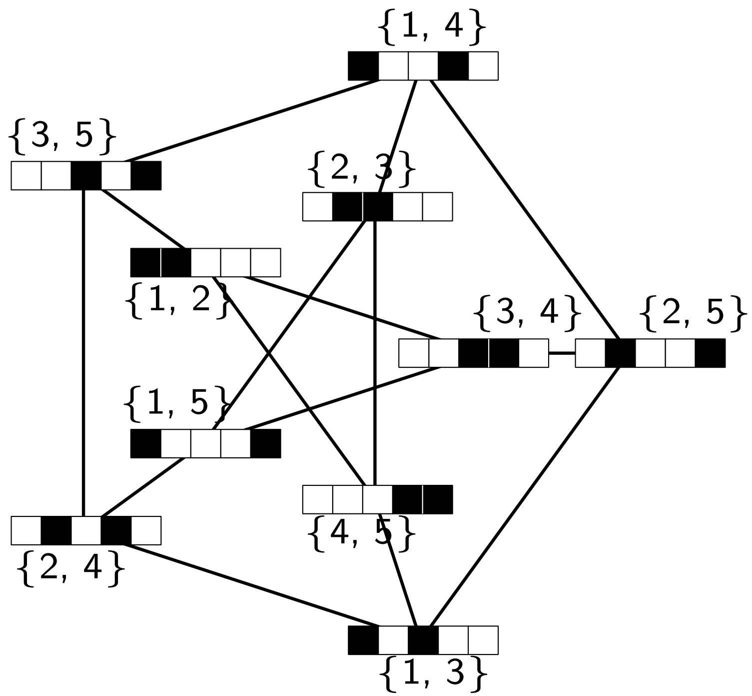

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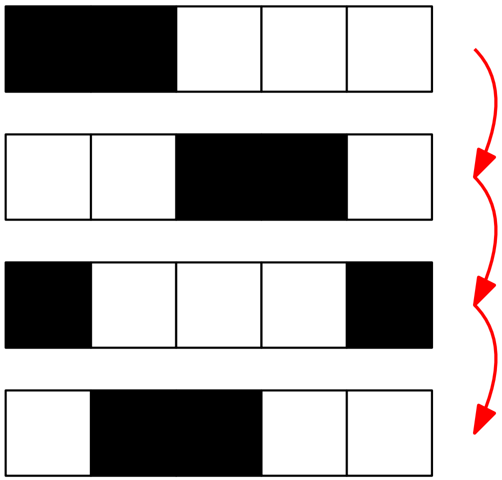

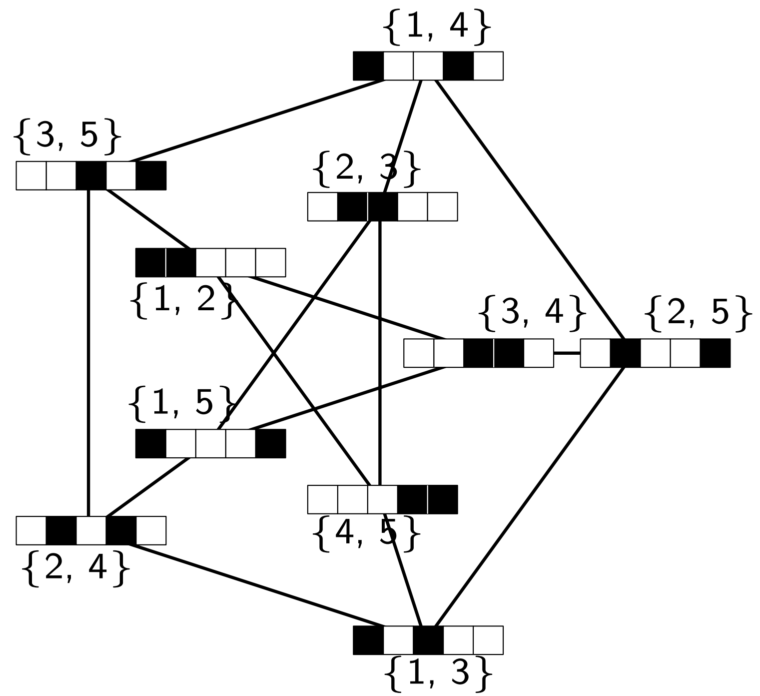

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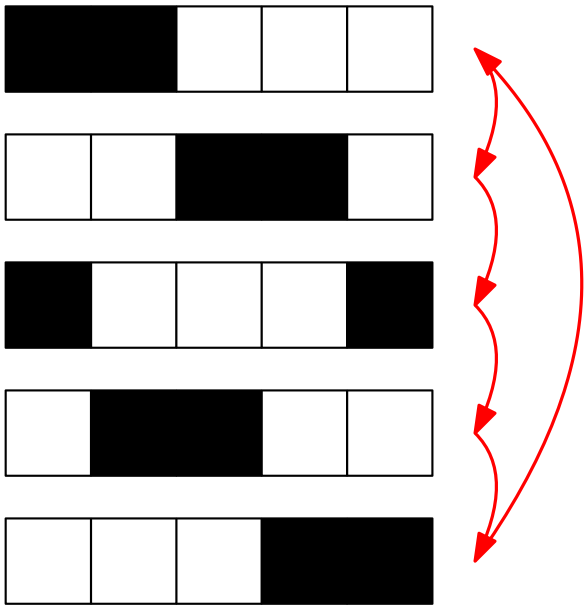

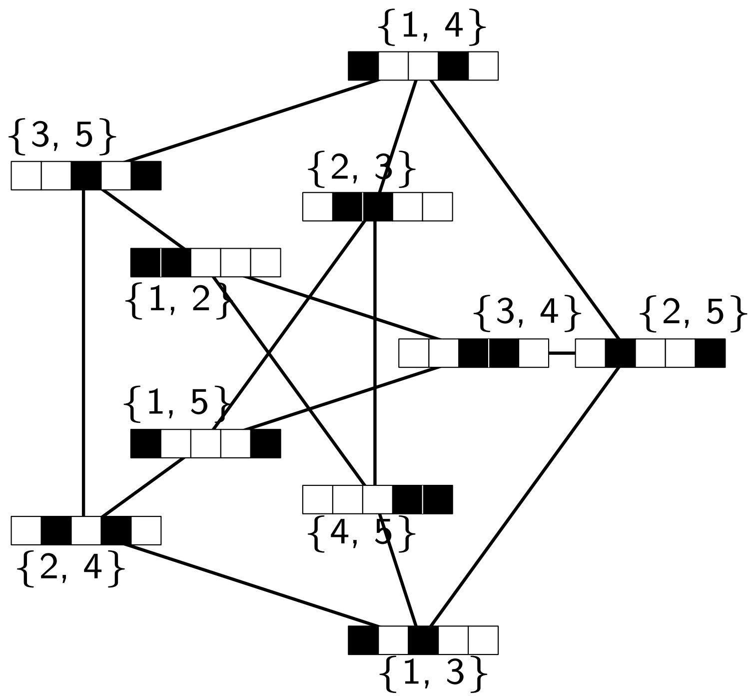

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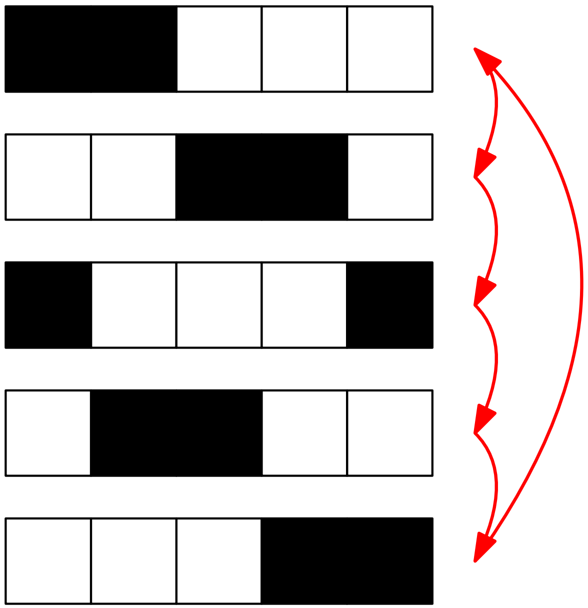

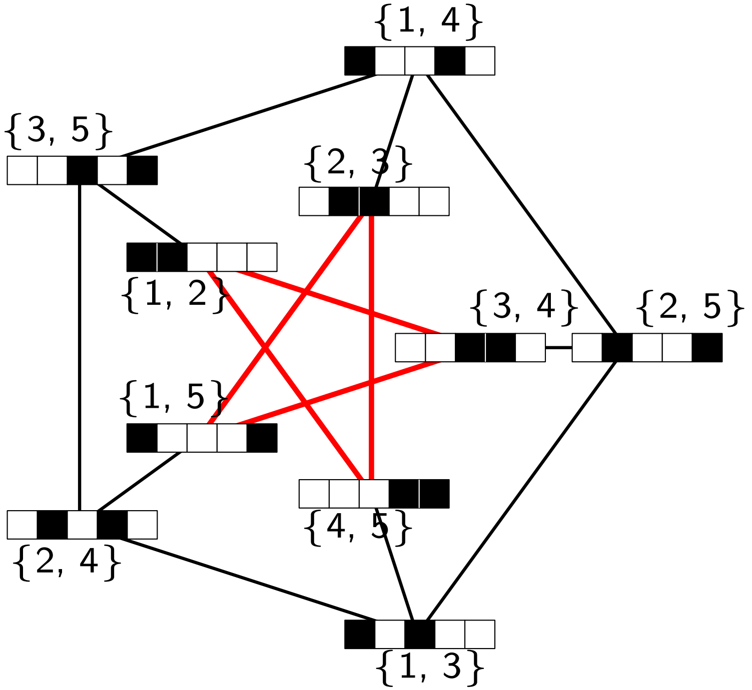

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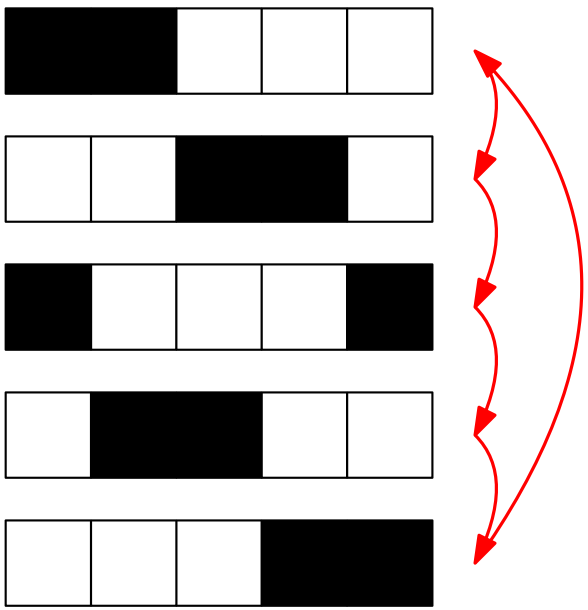

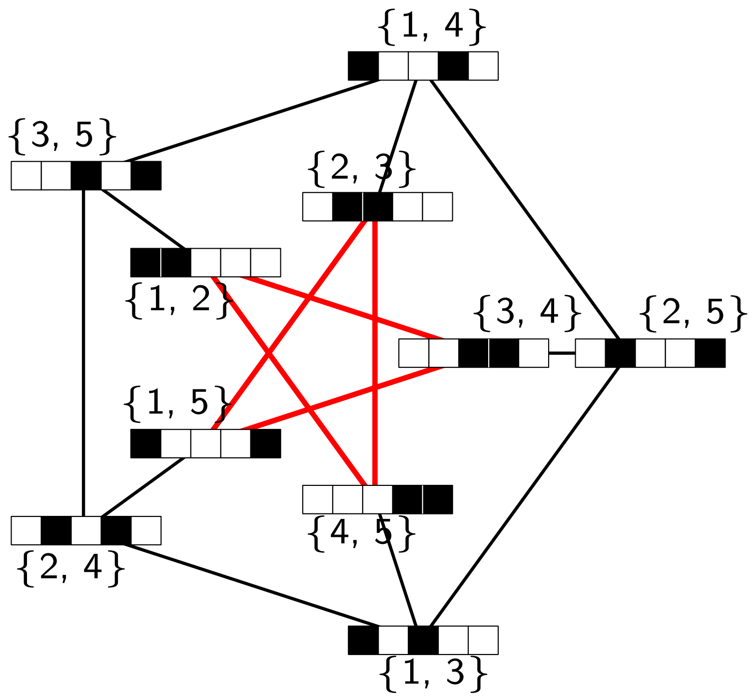

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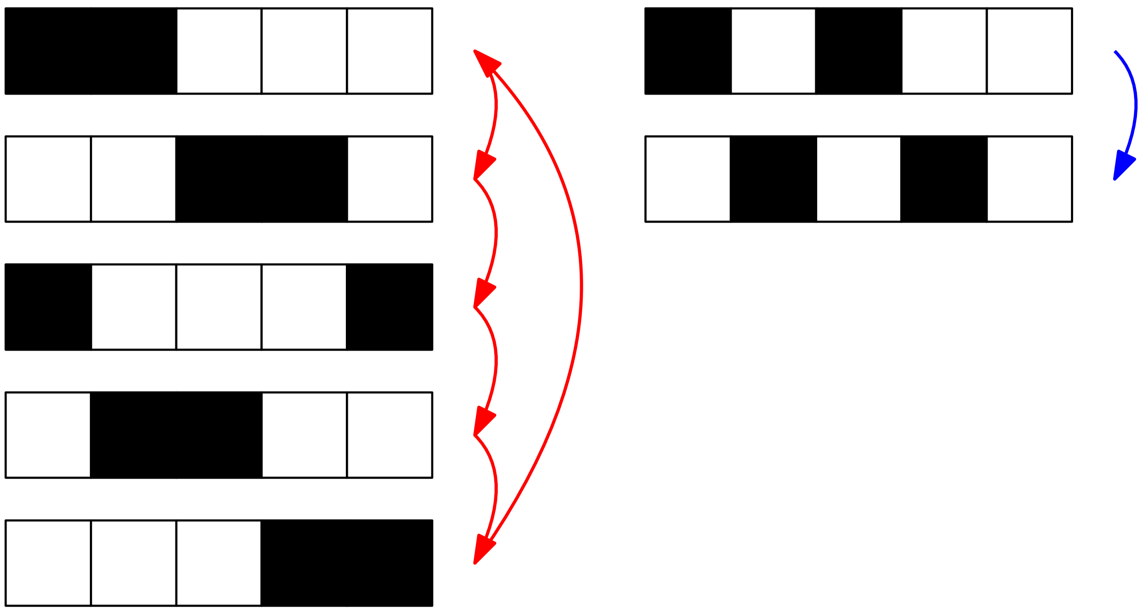

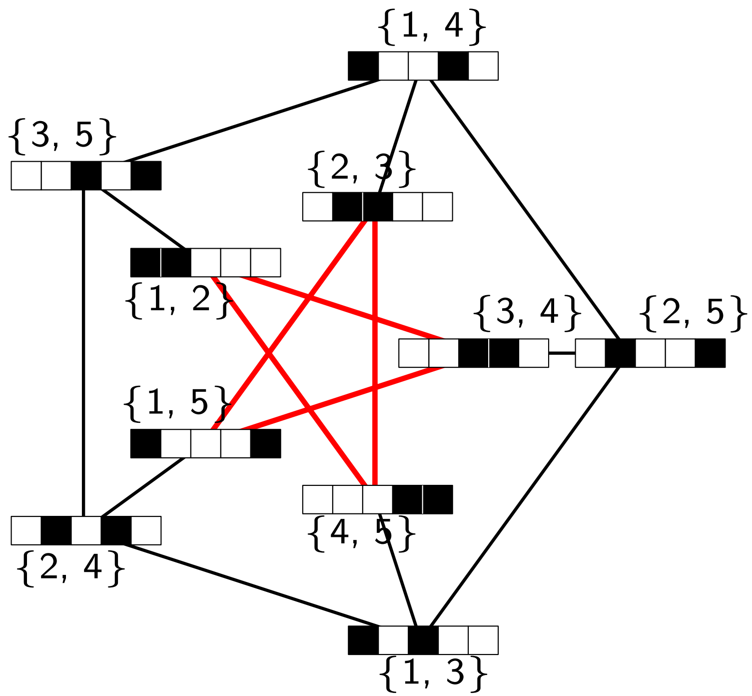

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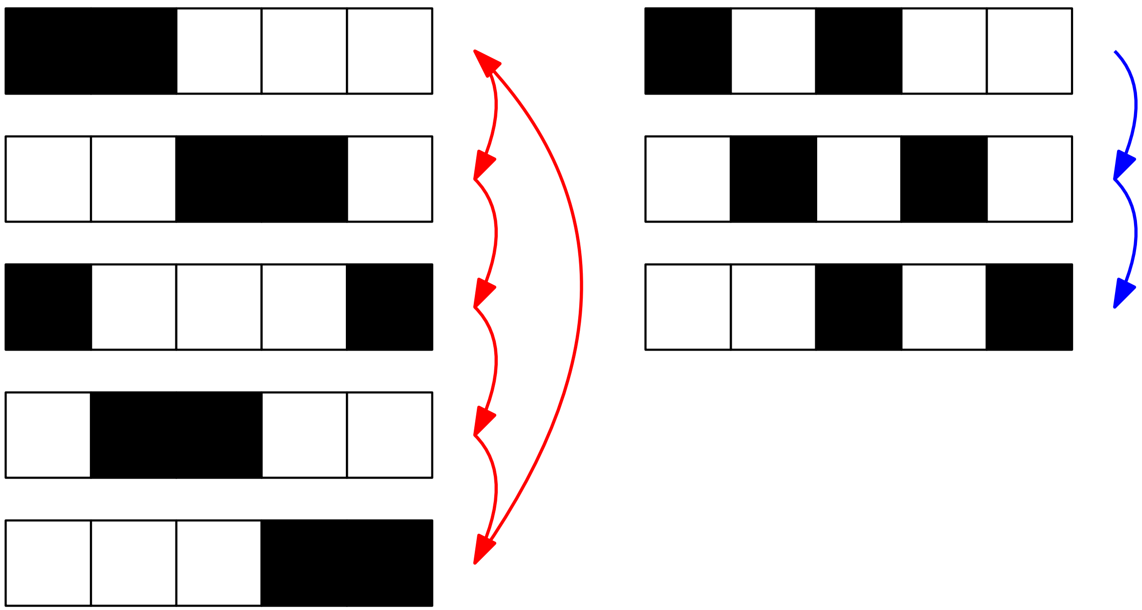

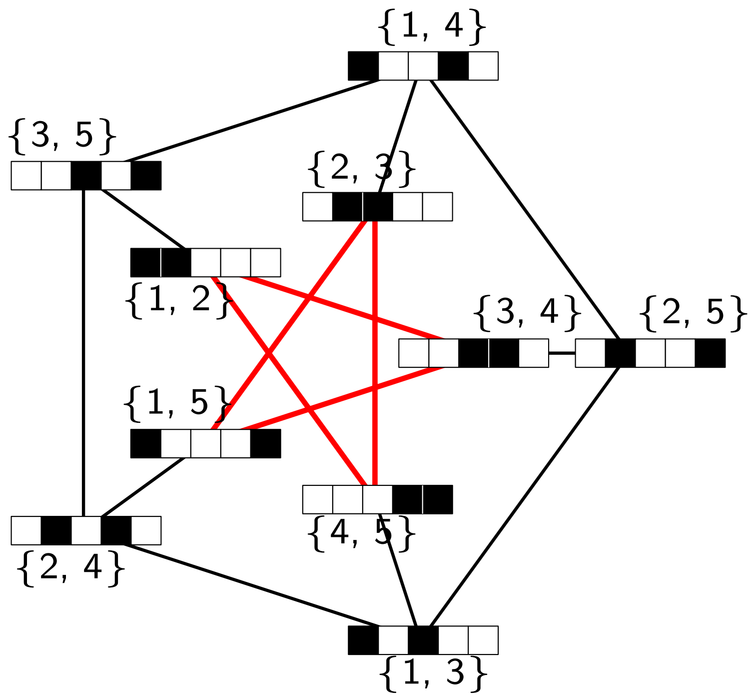

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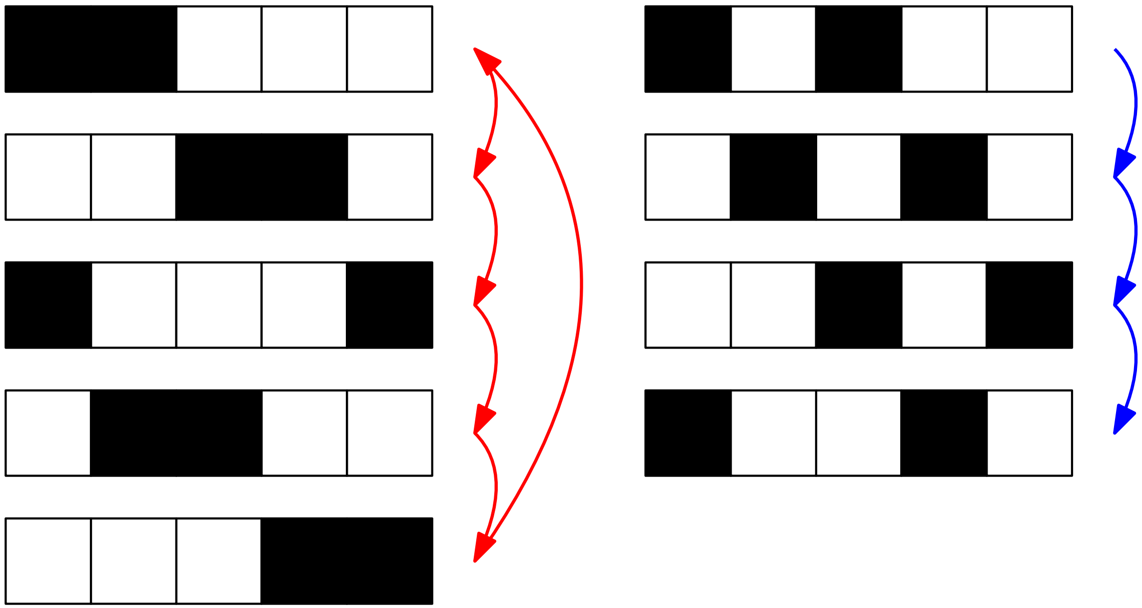

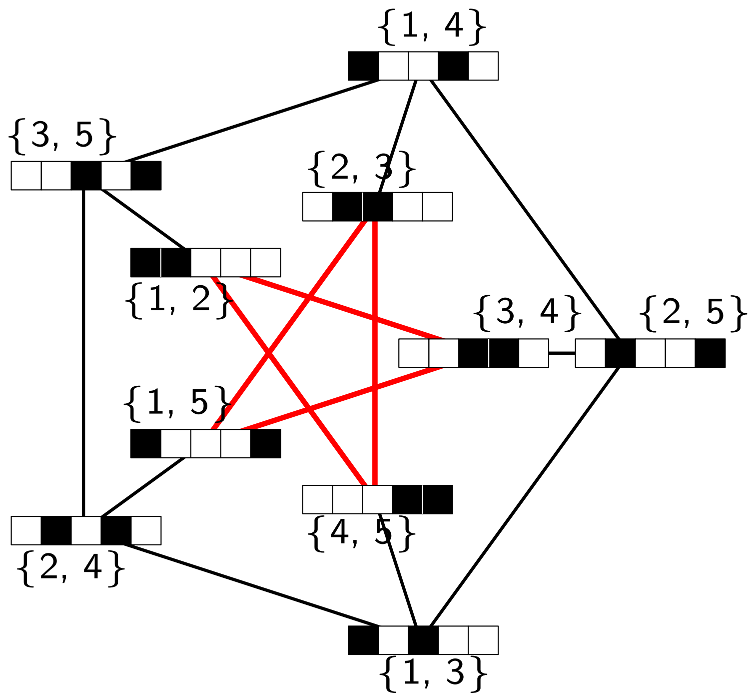

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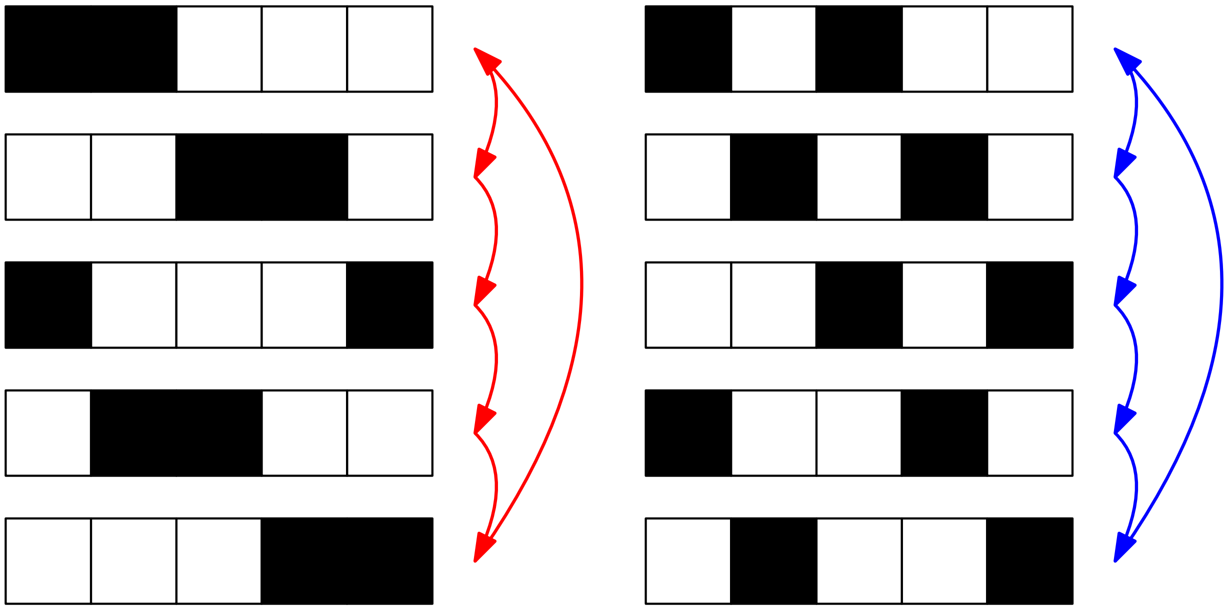

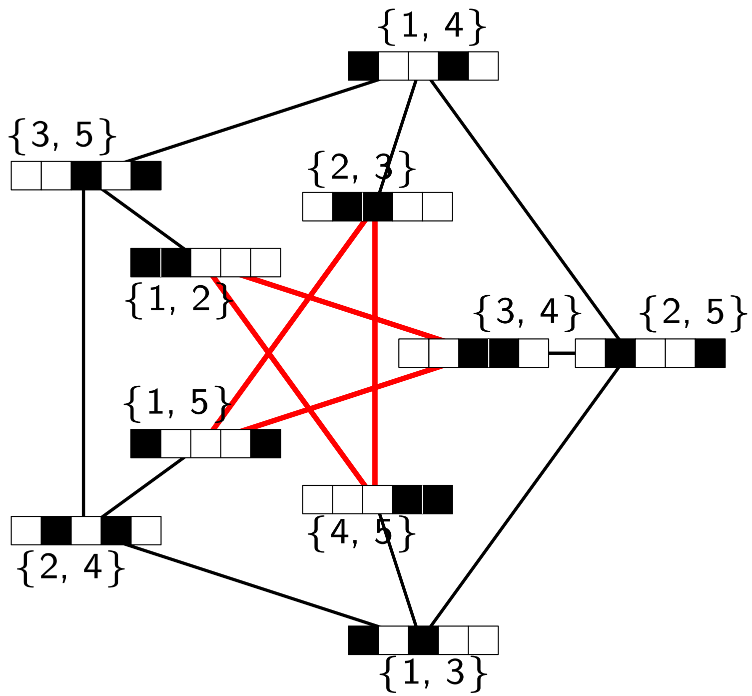

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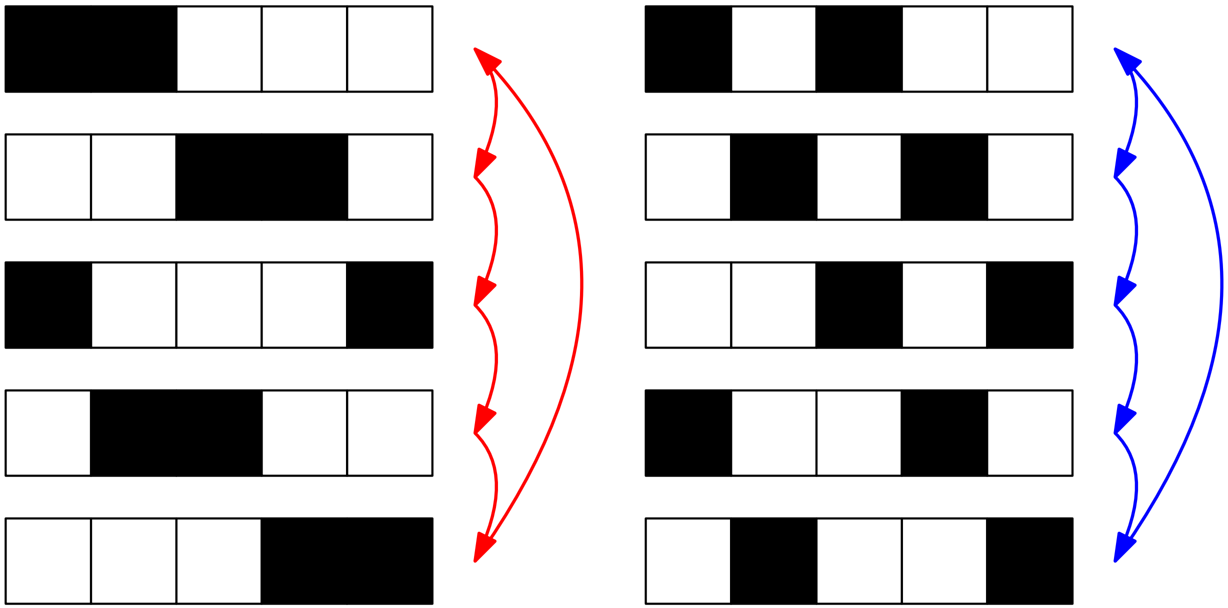
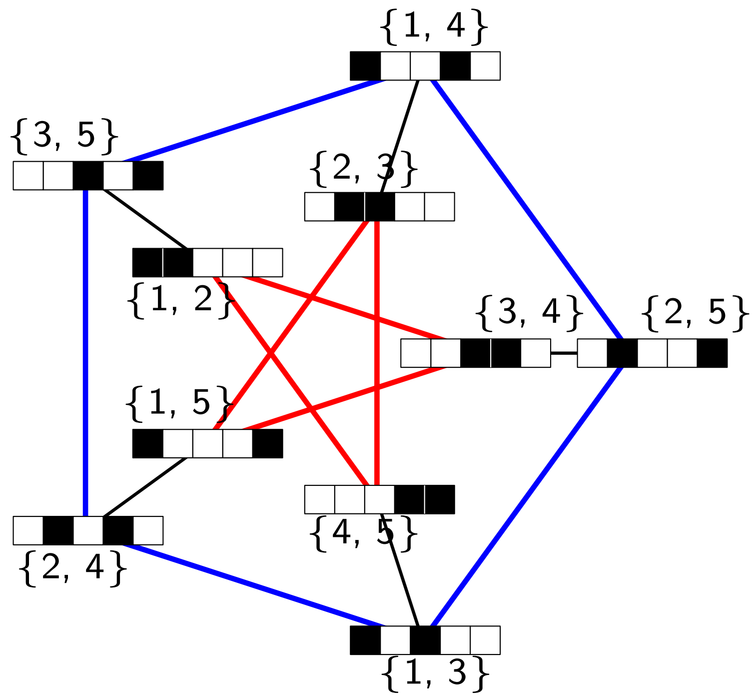
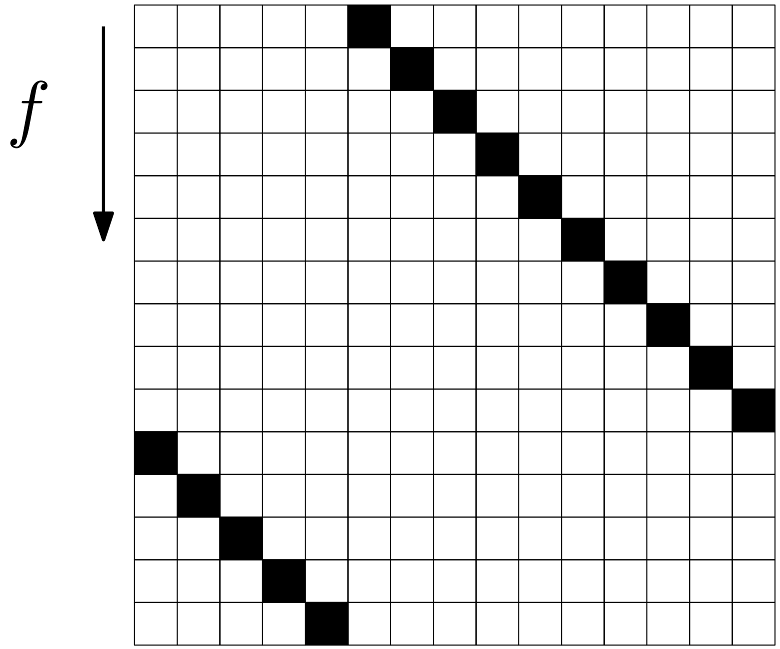




Analyzing the cycles

$(n, k) = (15, 1)$



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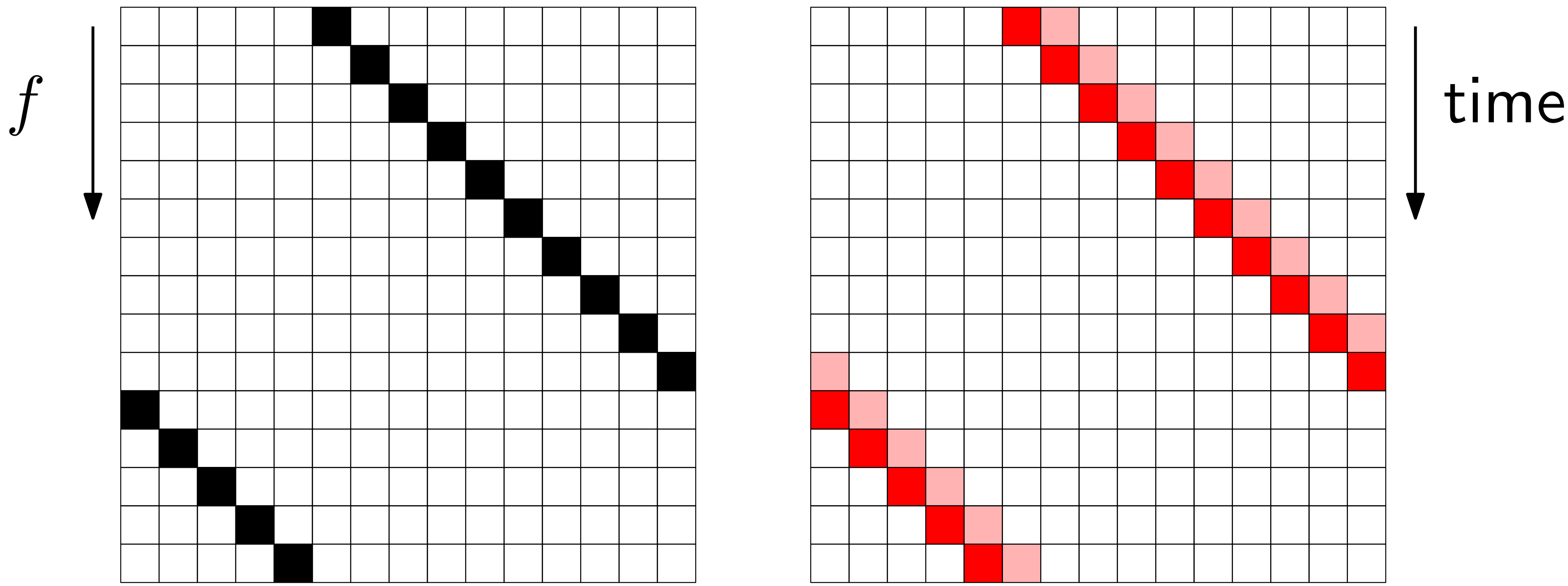

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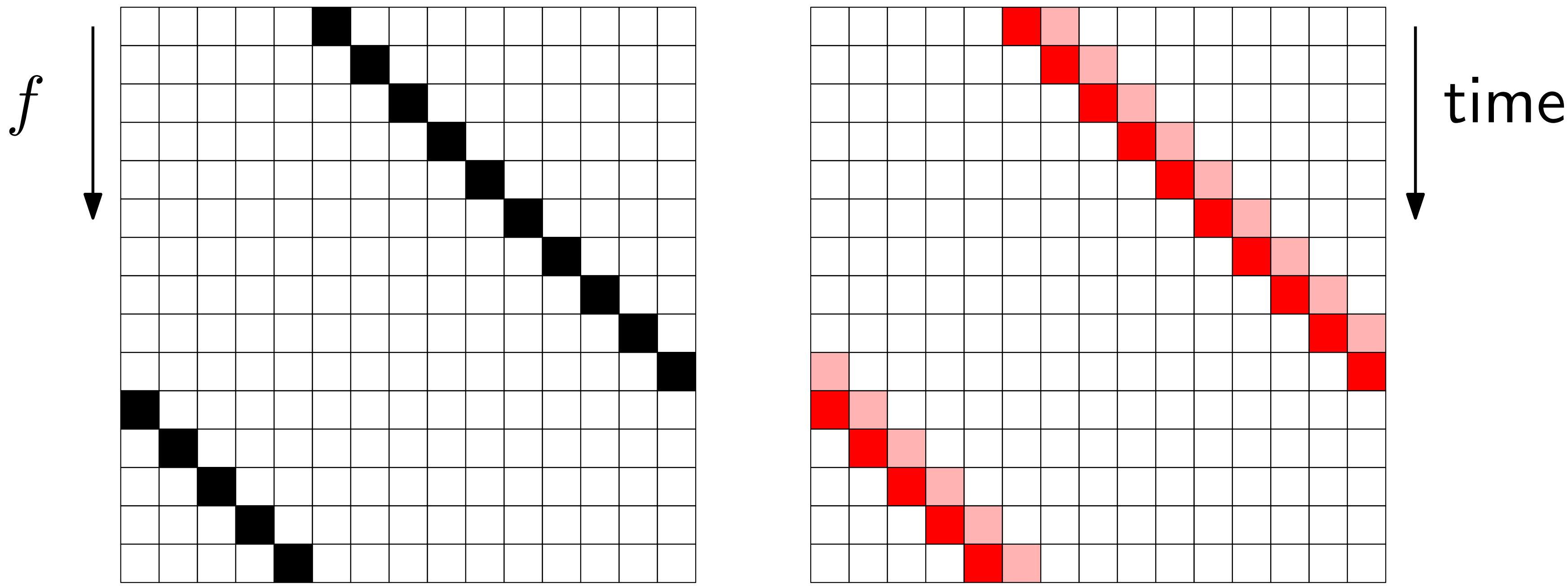
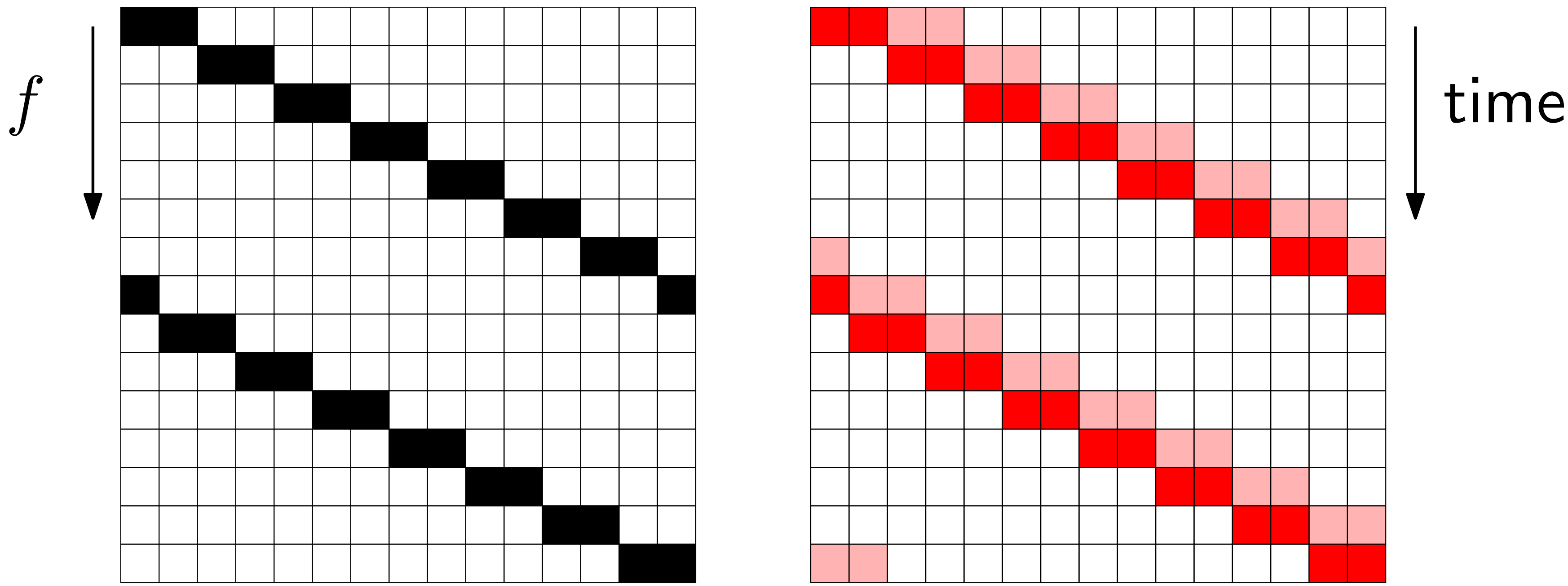


- Two matched bits form a glider
- Glider moves forward by 1 unit per step

Analyzing the cycles

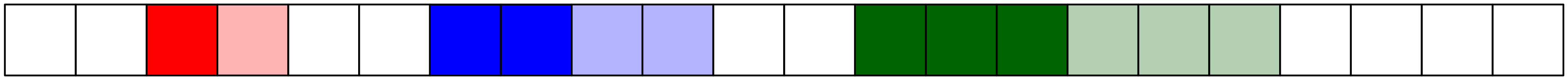
$(n, k) = (15, 2)$



- Four matched bits form one glider
- Glider moves forward by 2 units per step

Gliders

- **glider** := set of matched 1s and 0s (same number of each)



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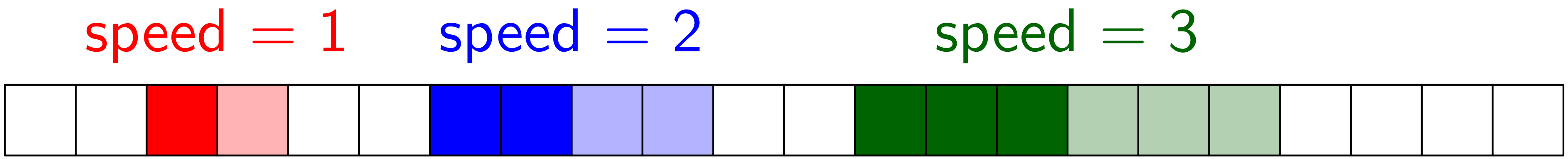

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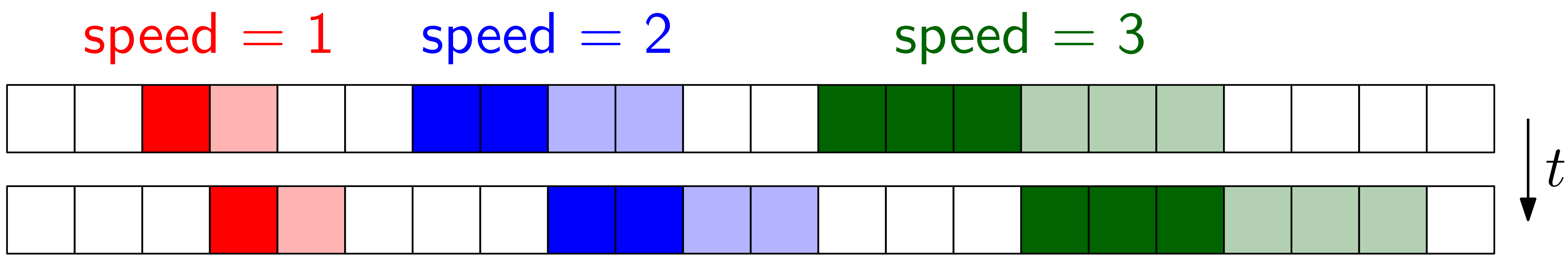

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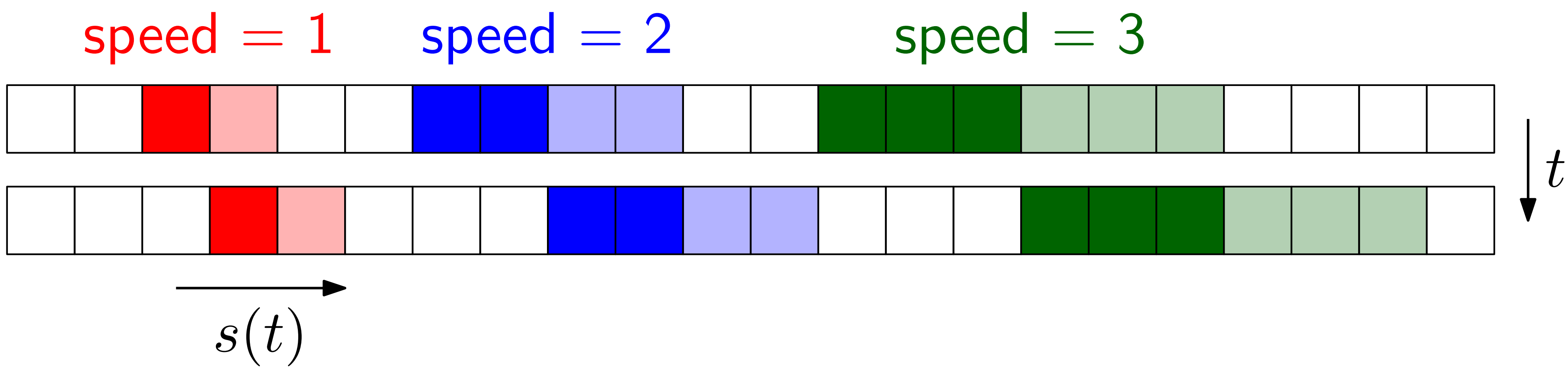

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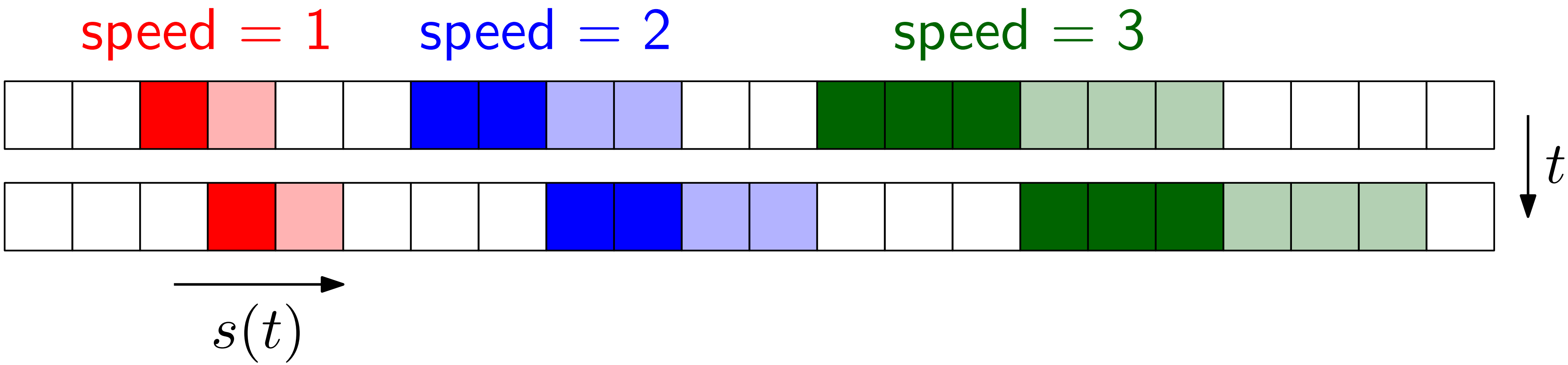

- Uniform equation of motion: $s(t) = v \cdot t + s(0)$

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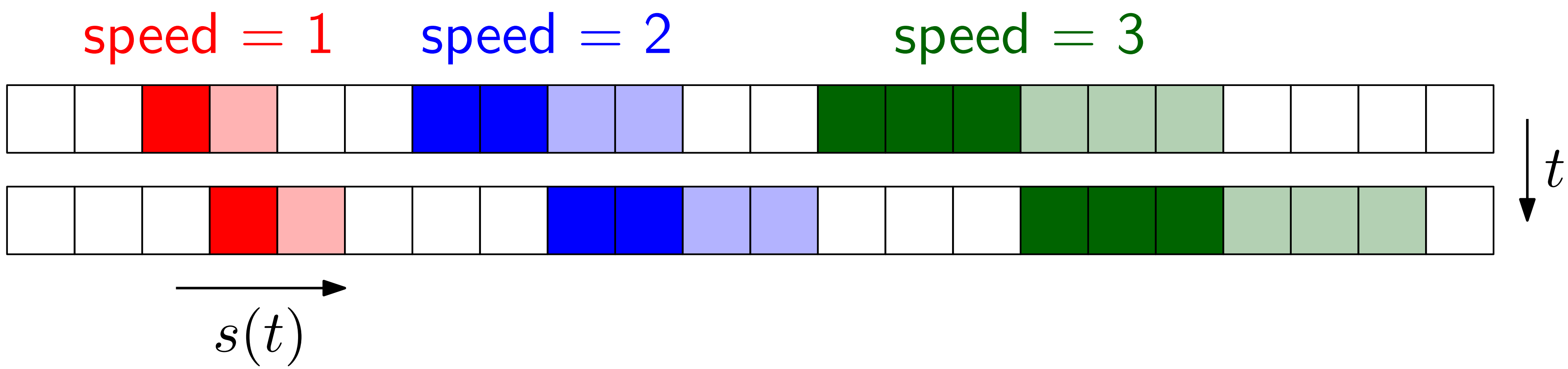


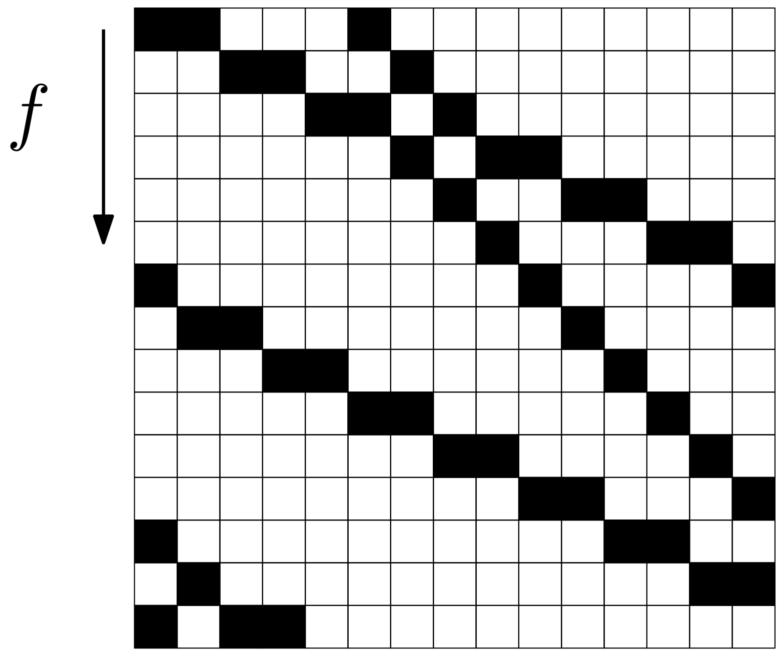
- Uniform equation of motion: $s(t) = v \cdot t + s(0)$

position (modulo n) speed

time t = number of applications of f starting position

Overtaking of gliders

$(n, k) = (15, 4)$



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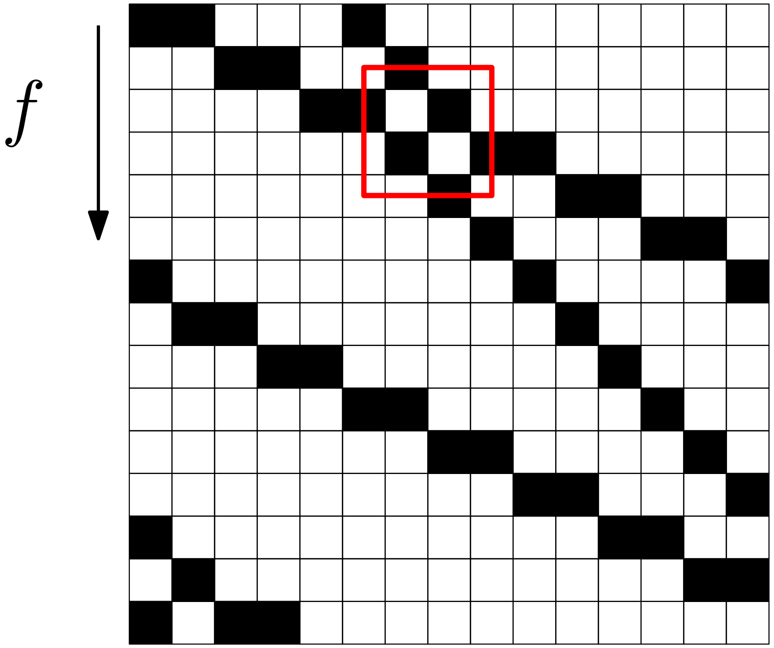

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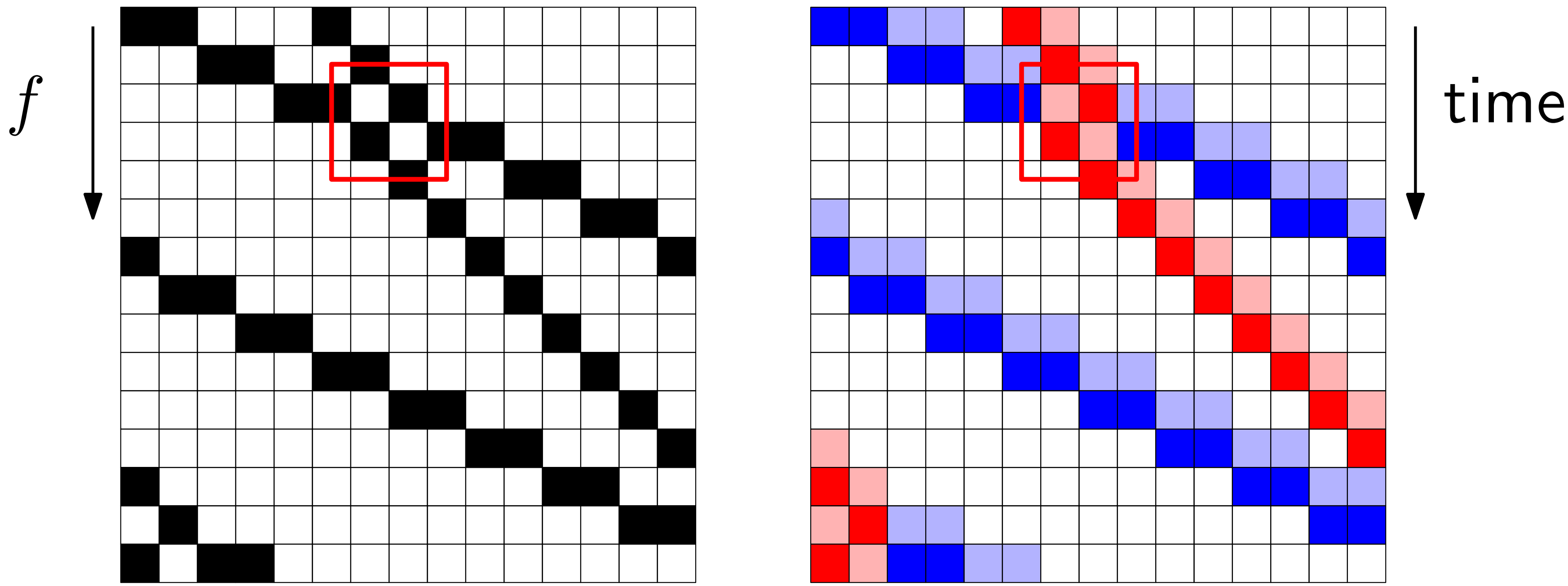

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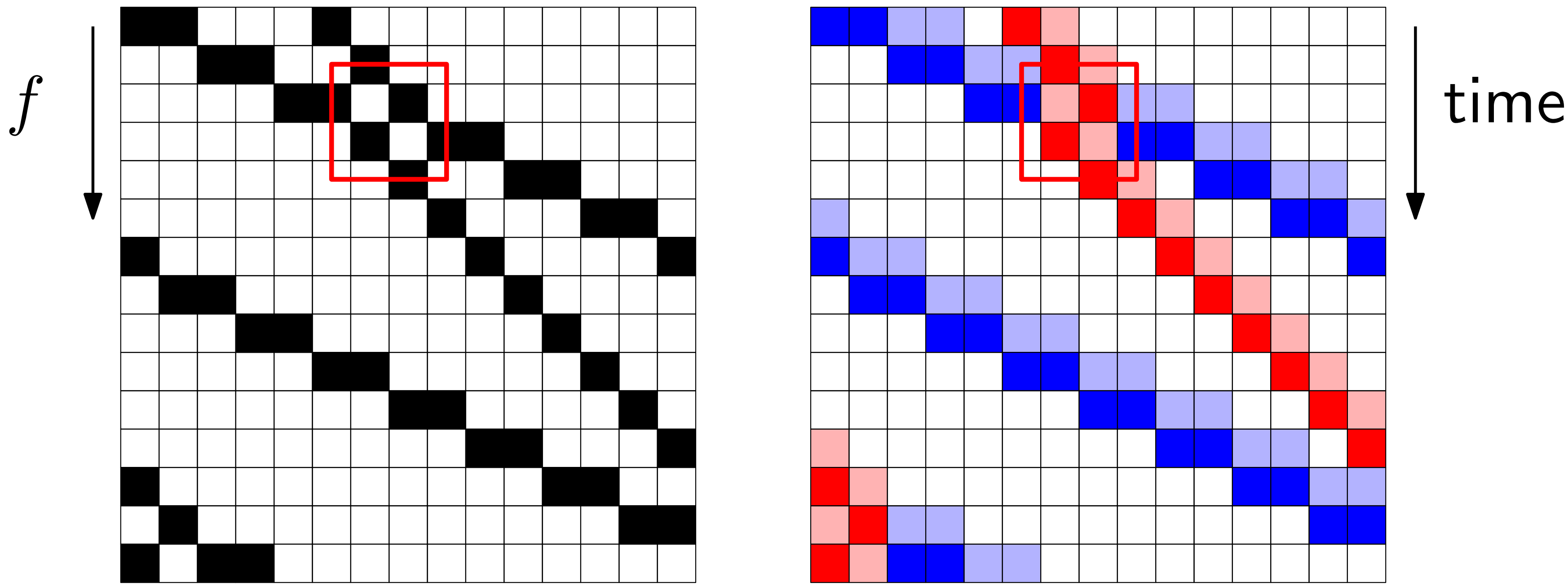

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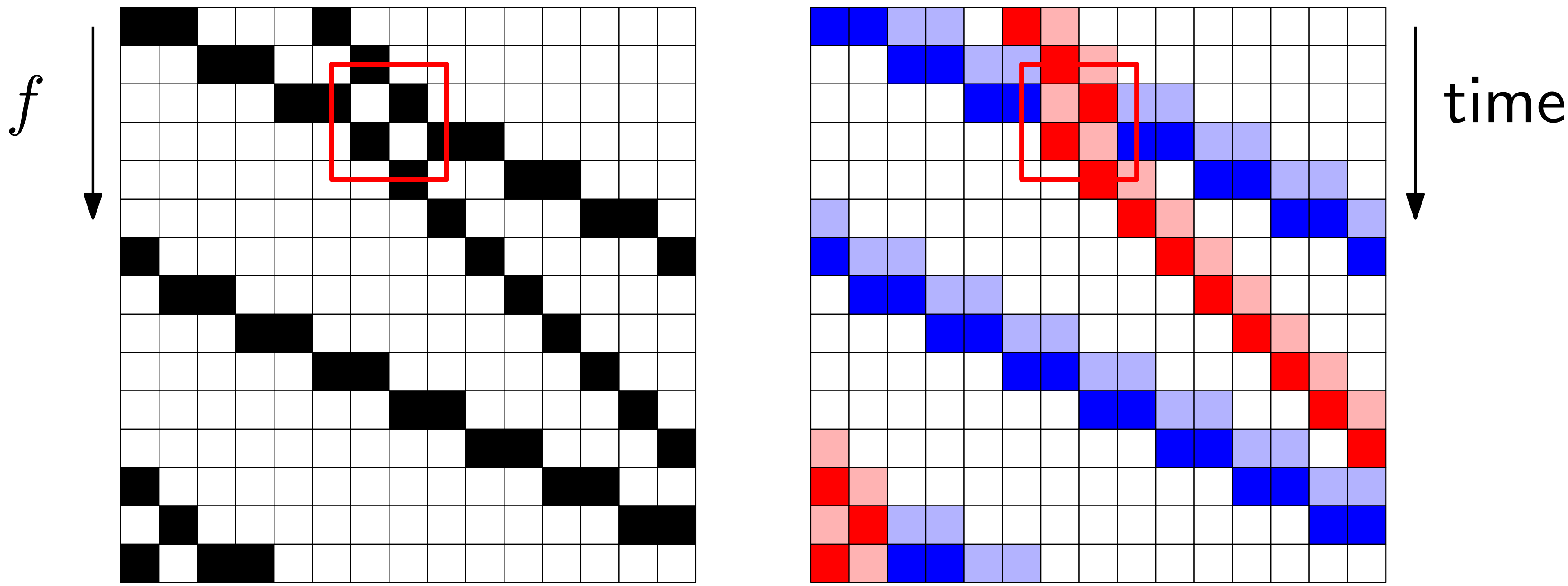
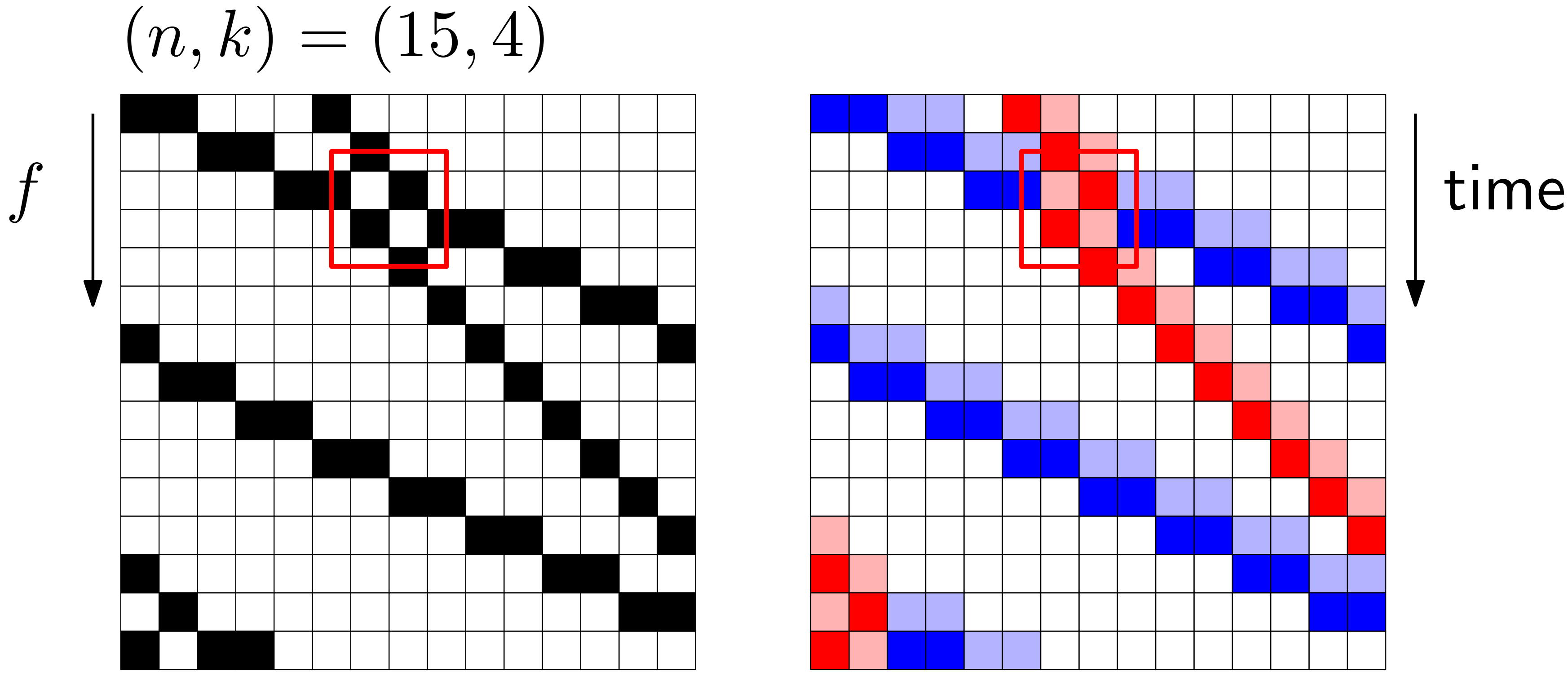


- during overtaking, slower glider stands still for two time steps
- faster glider is boosted by twice the speed of slower glider

Overtaking of gliders



- non-uniform equations of motion:

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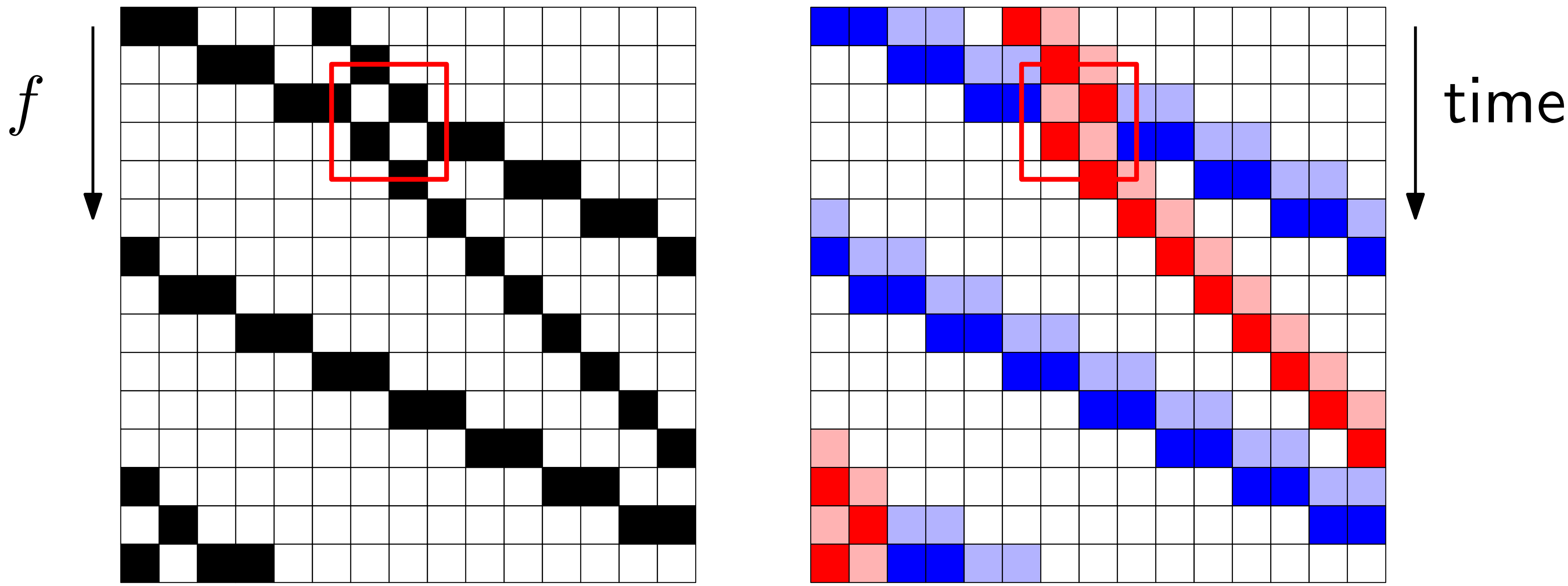

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$$s_1(t) = v_1 \cdot t + s_1(0) \color{red}{-2v_1 \cdot c_{1,2}}$$
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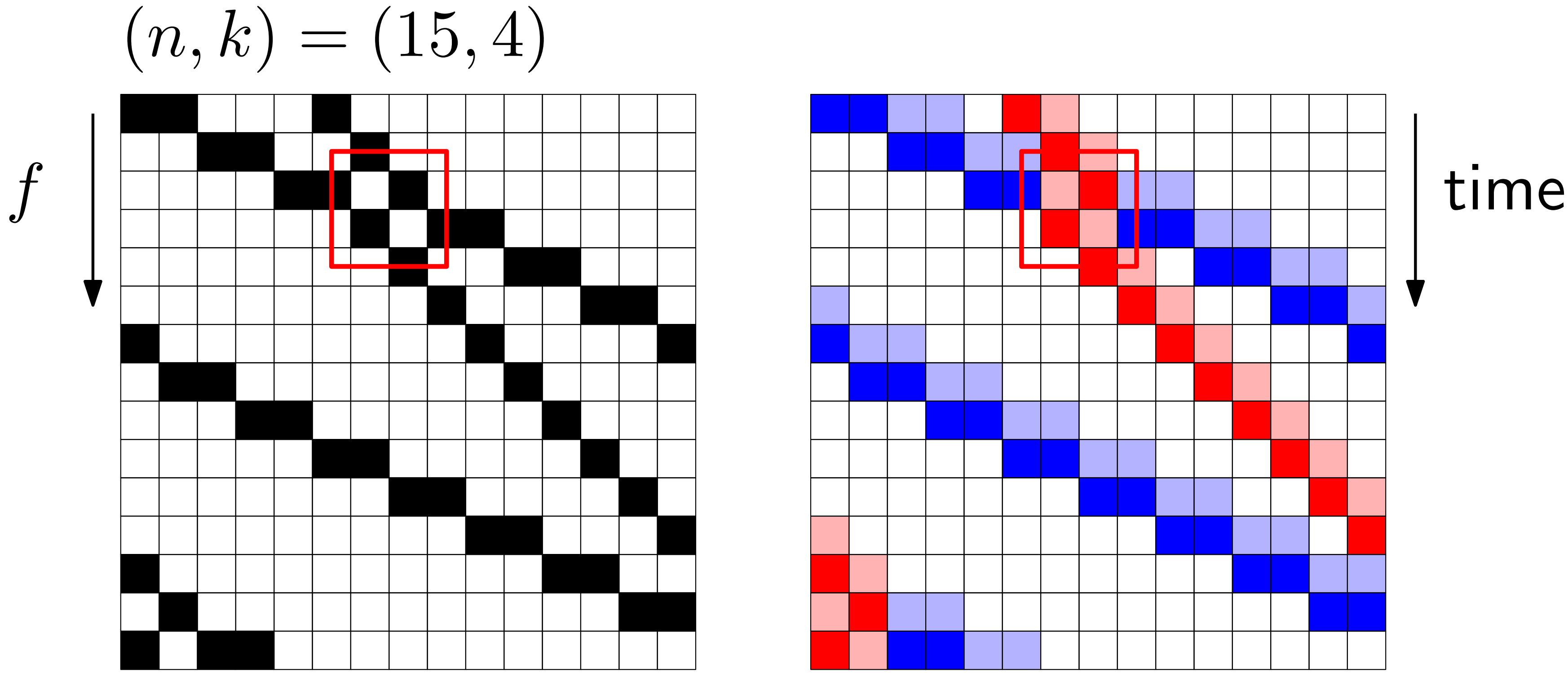

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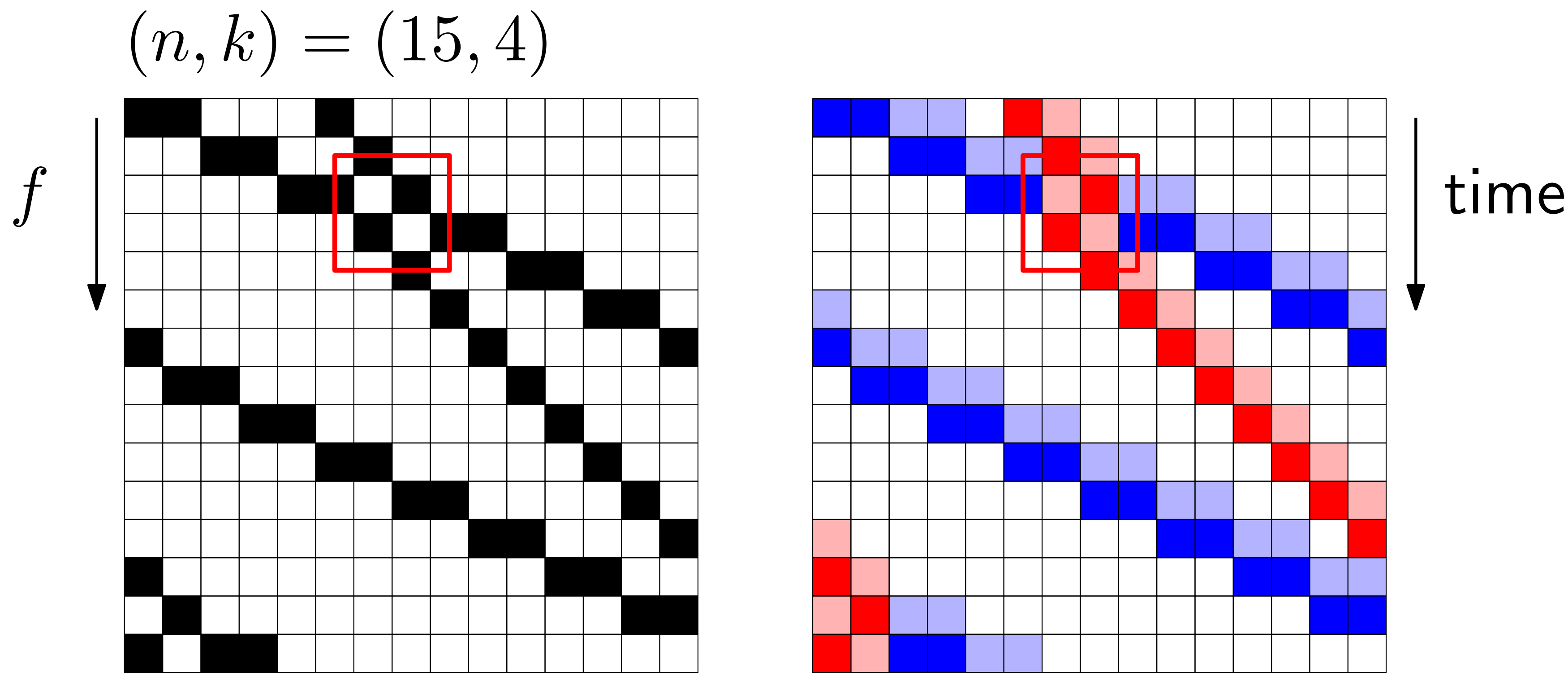
$c_{1,2}$:= number of overtakings

Overtaking of gliders



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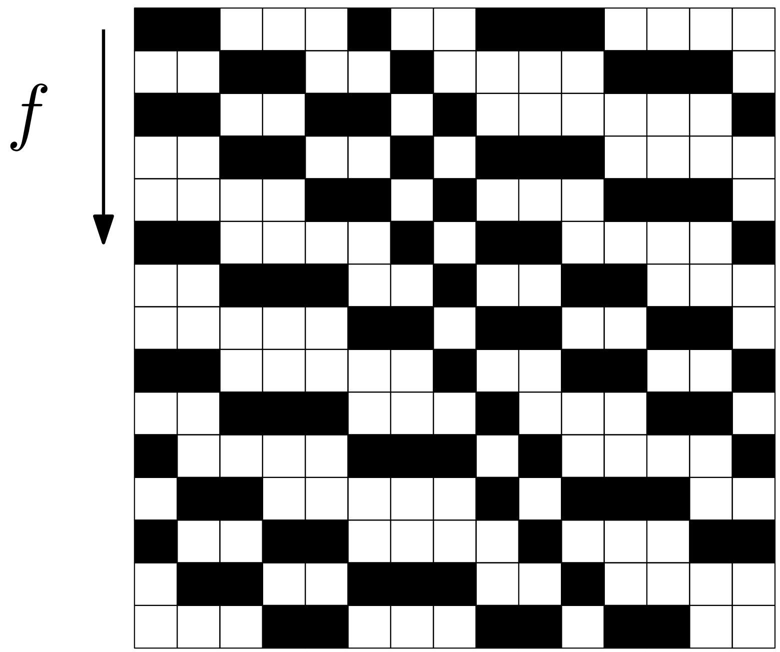
energy conservation!

$c_{1,2} :=$ number of overtakings

Glider partition

$$(n, k) = (15, 6)$$



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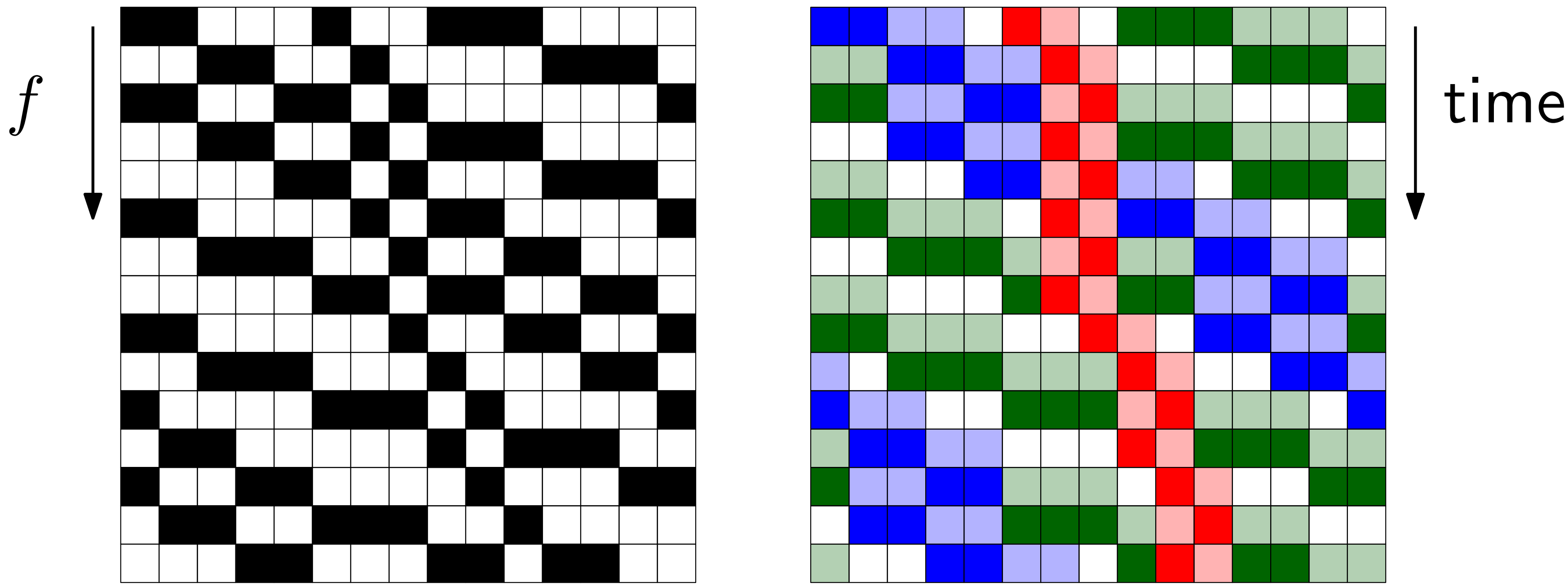

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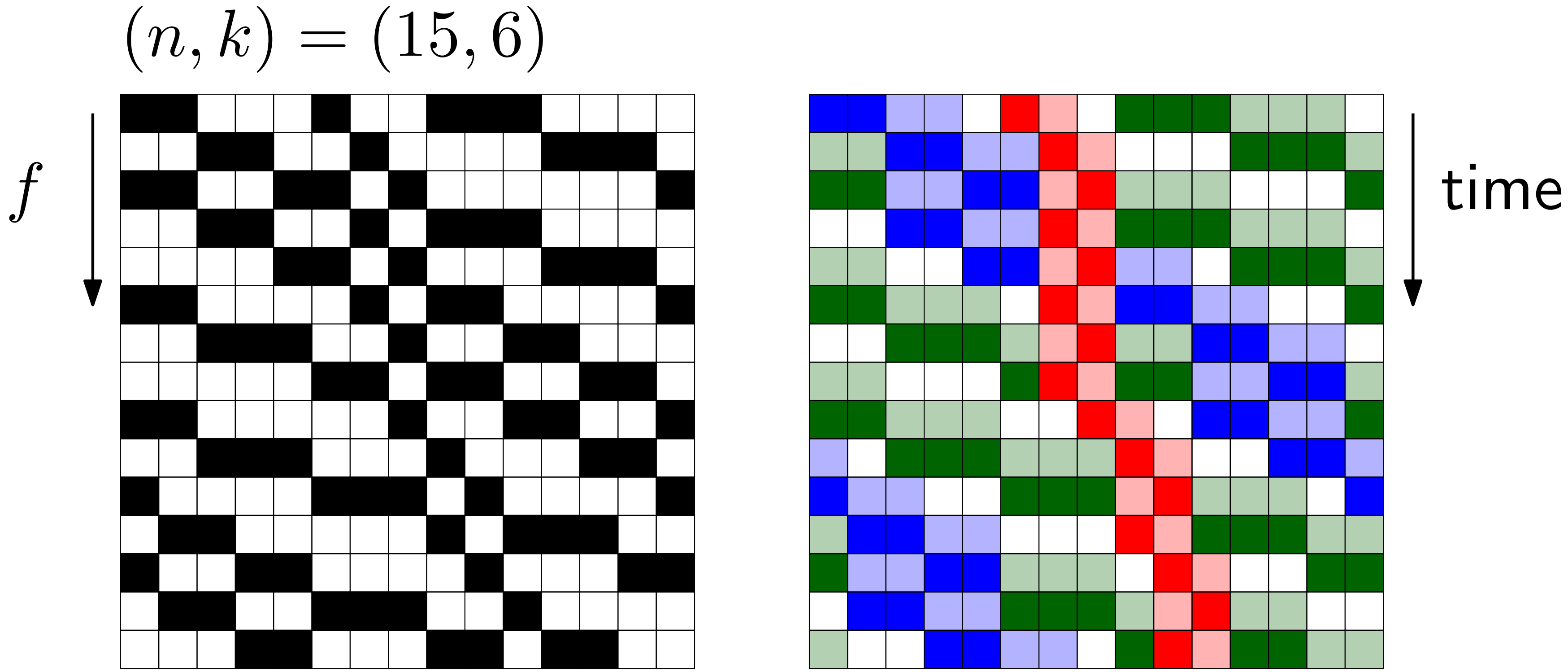

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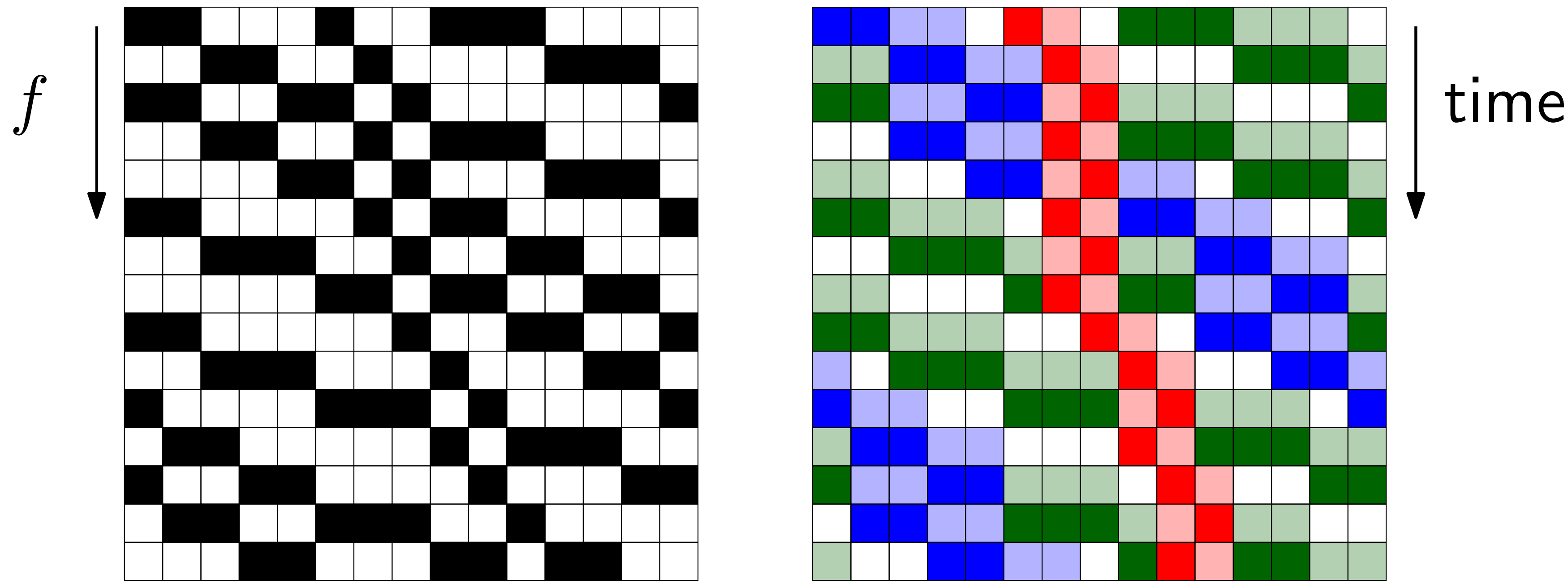

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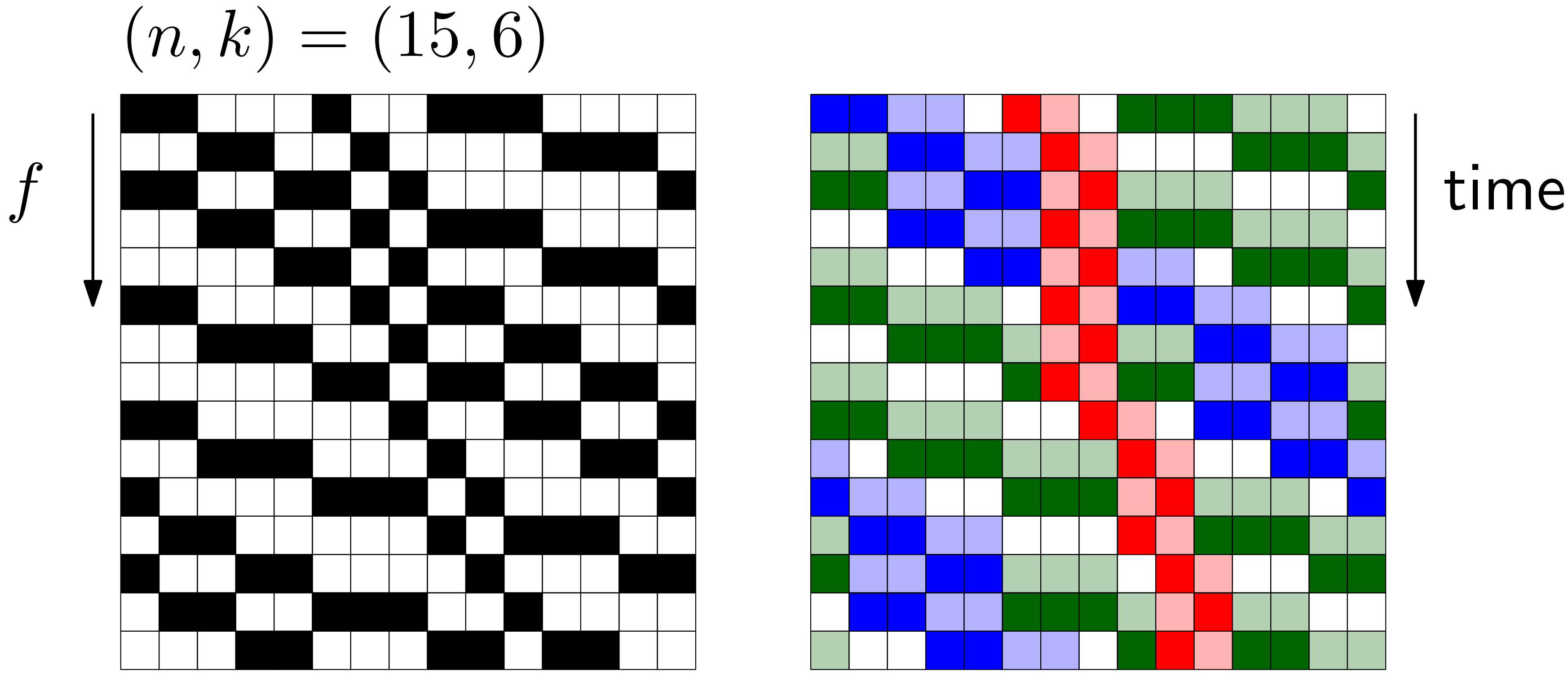

- gliders can be interleaved in complicated ways
- general glider partition rule works recursively on Motzkin path
- general equations of motion have overtaking counters $c_{i,j}$ for all pairs of gliders i, j

Cycle invariant

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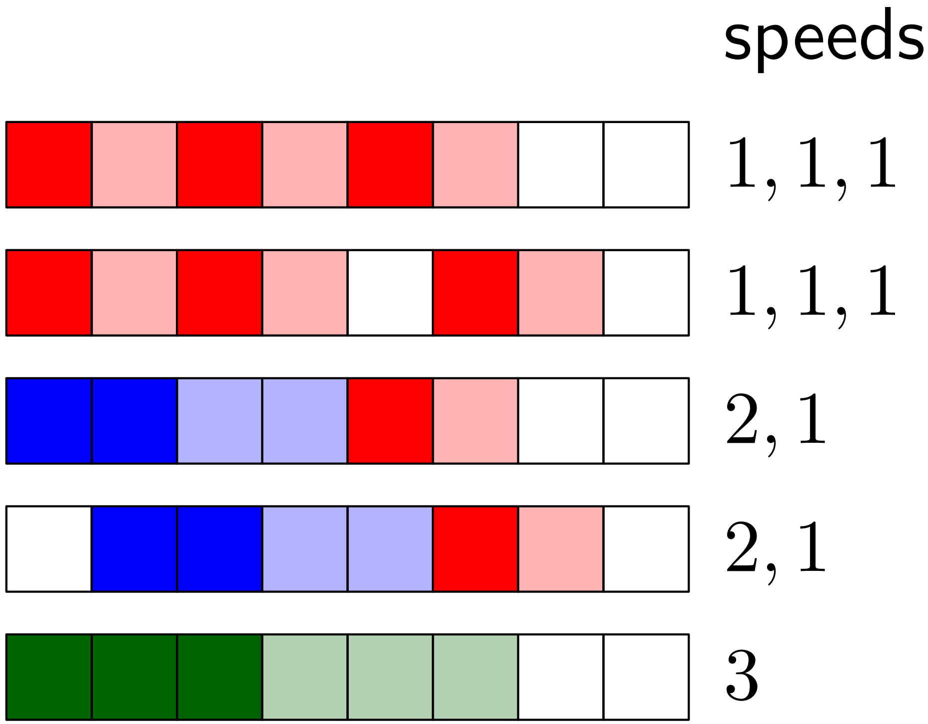

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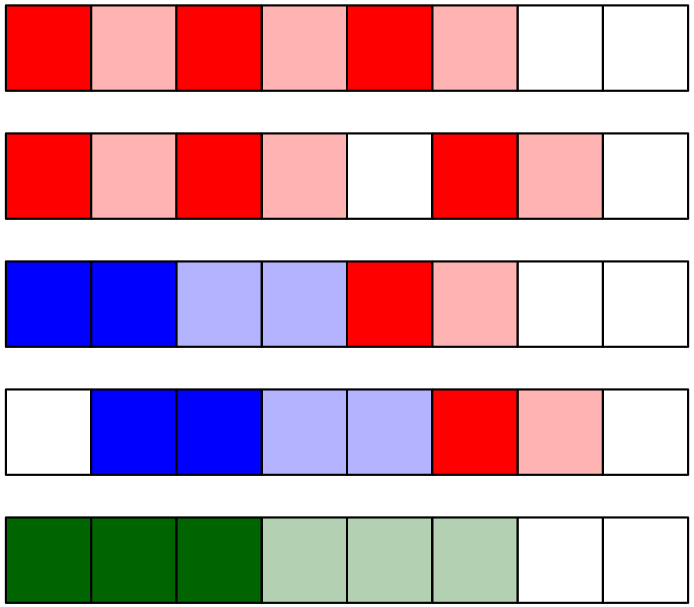

speeds	cycle length
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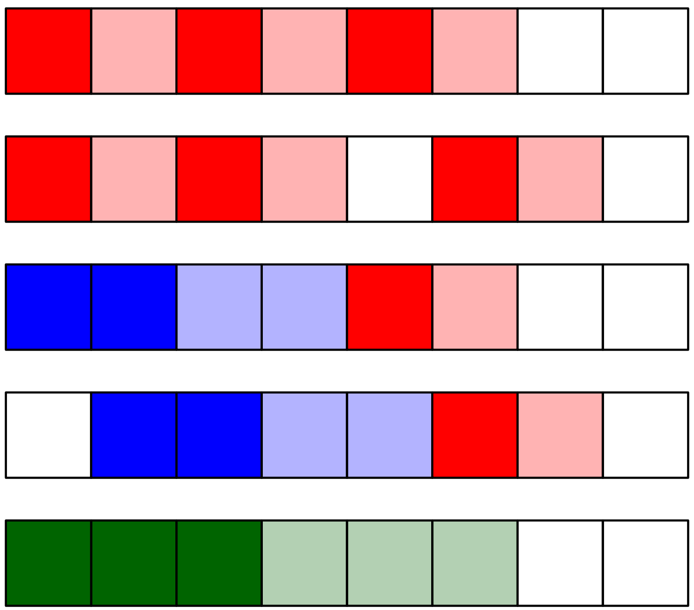

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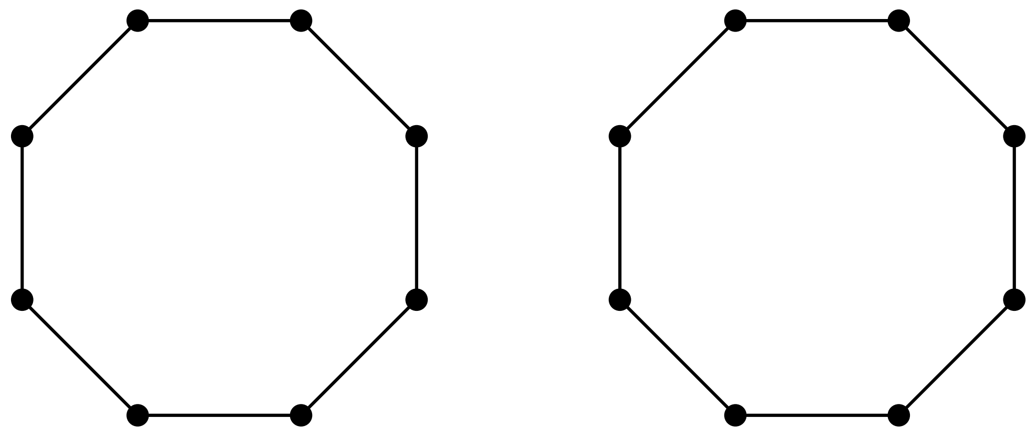
Gluing cycles



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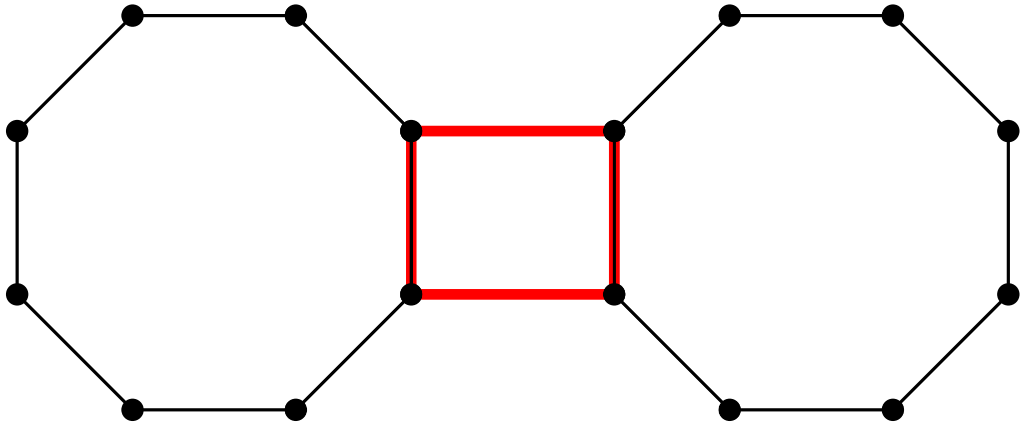

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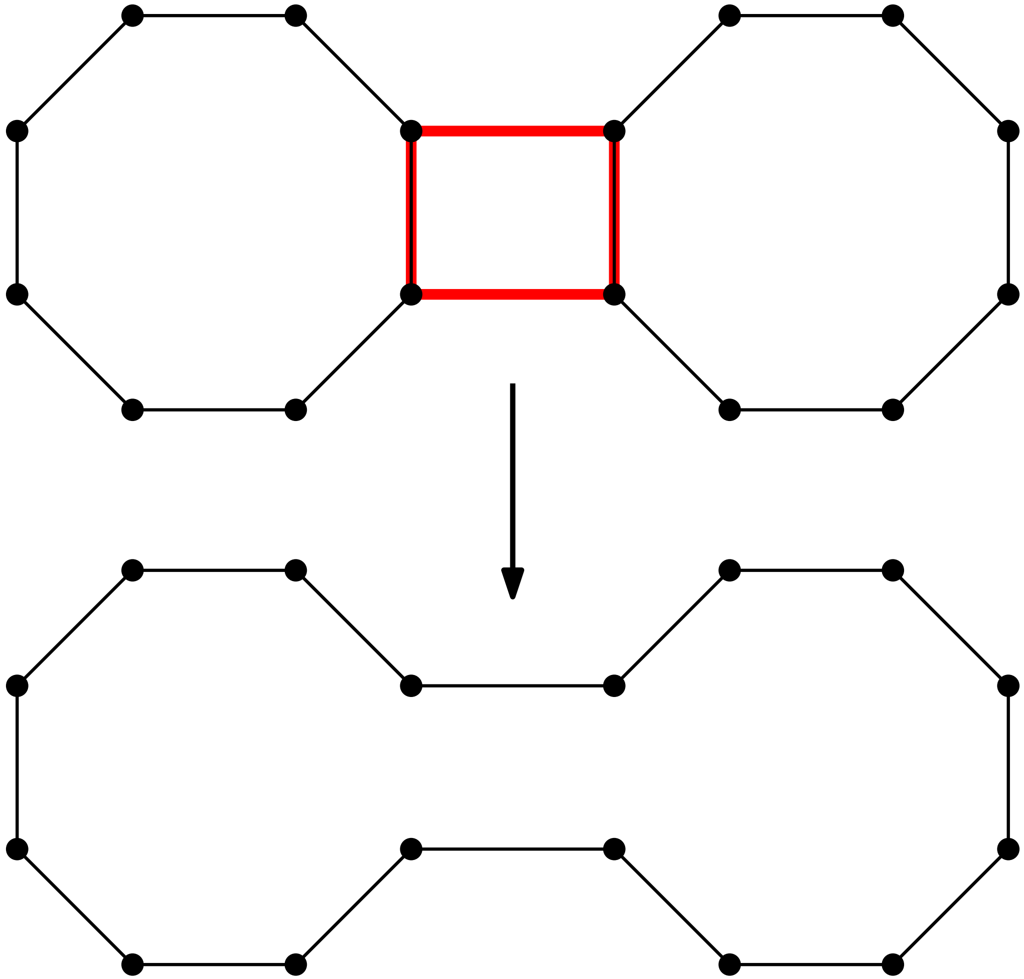

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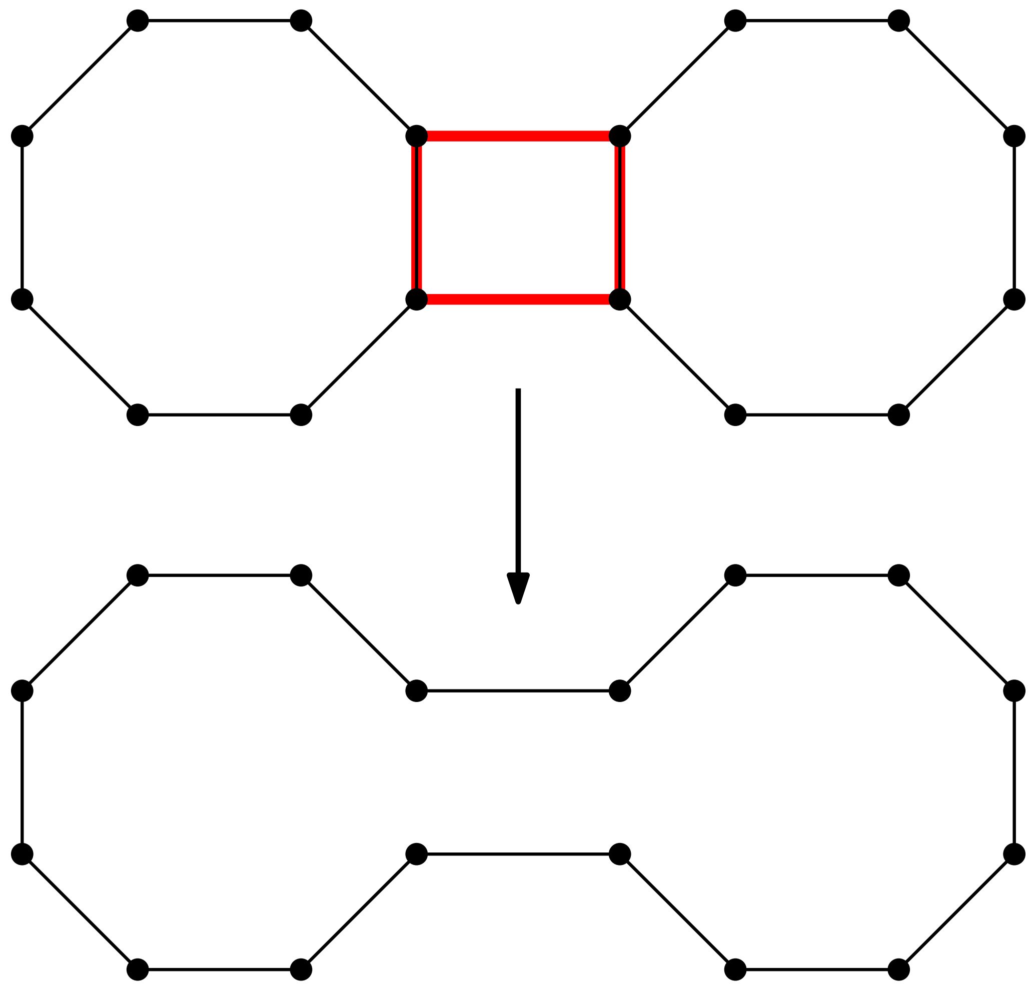
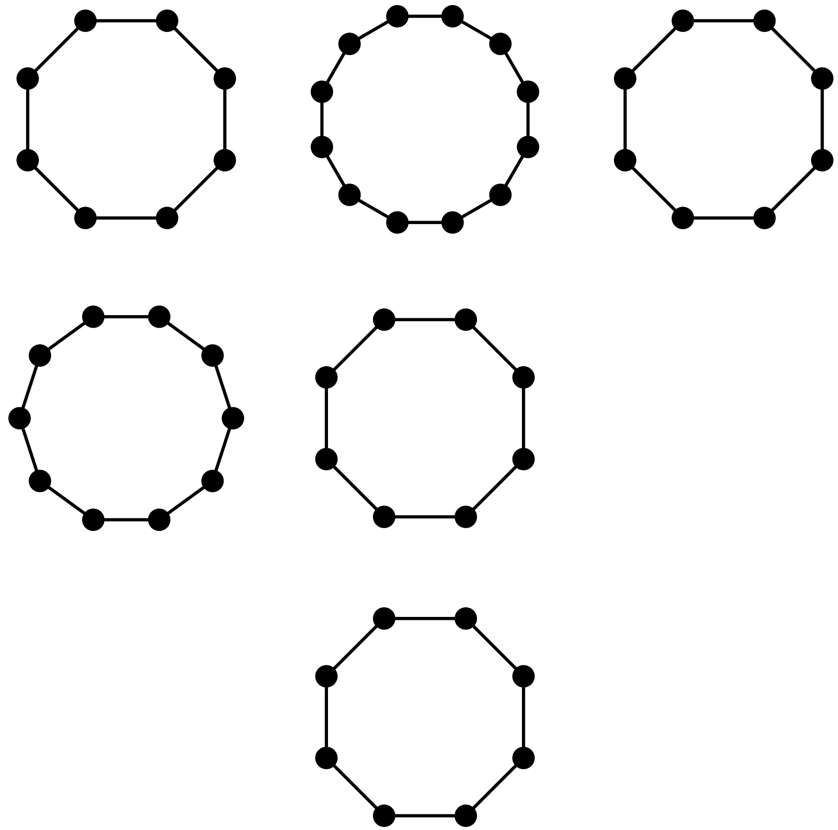


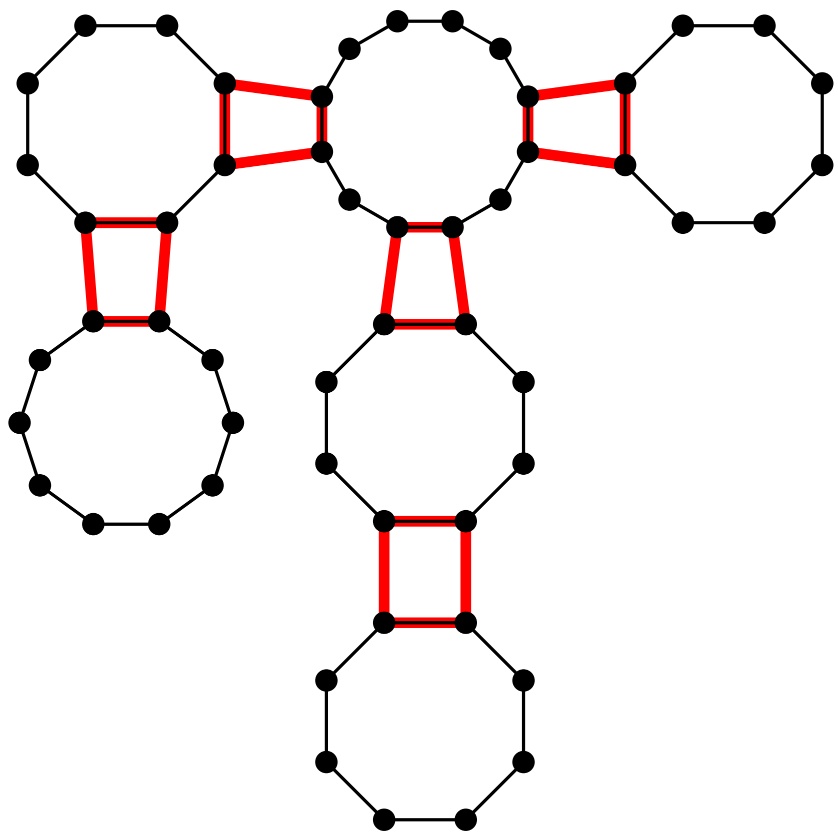
4-cycles exist as $n \geq 2k+3$

Gluing cycles



Gluing cycles

- connect cycles of factor to a single Hamilton cycle (tree-like)



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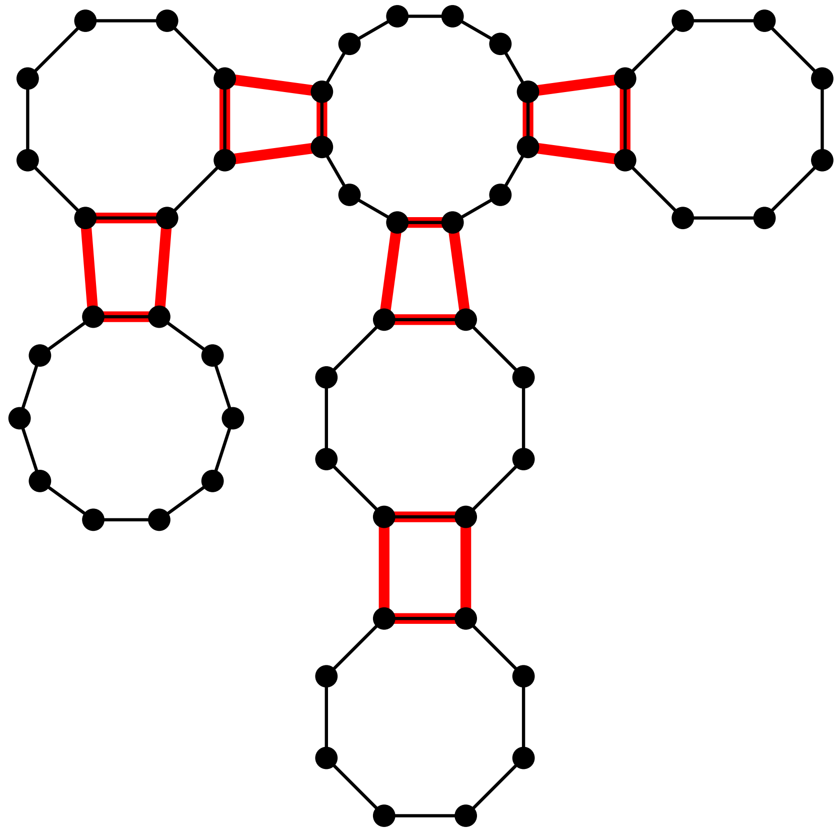

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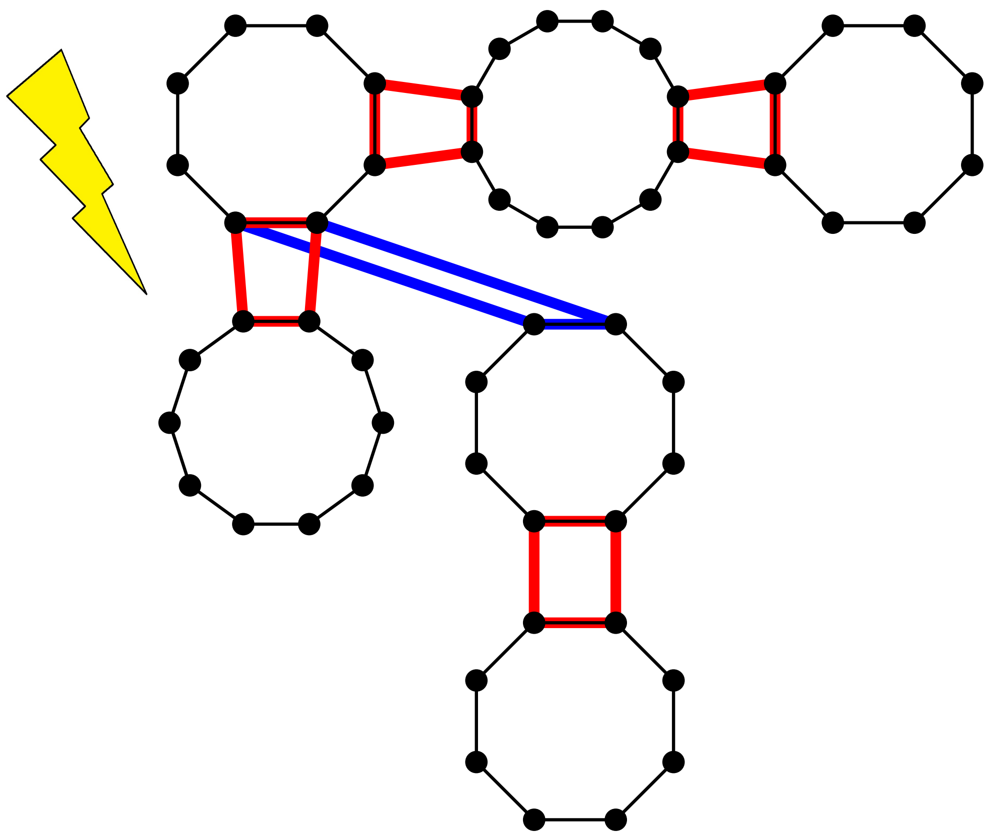

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- **Conjecture** [Ruskey, Savage 1993]: Does every matching of Q_n extend to a Hamilton cycle?

Thank you!