Intersecting Density of Permutation Groups

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Permutations

Definition

Two permutations $\sigma, \pi \in \mathrm{Sym}(n)$ **agree or intersect** if for some $i \in \{1, 2, \ldots, n\}$

$$i^\sigma = i^\pi.$$

A **set** of permutations from a group G is **intersecting** if any two elements from the set are intersecting.

Example

The following 17 permutations from $\mathrm{Sym}(5)$ that are intersecting

$$(), \quad (4,5), \quad (1,4), \quad (1,5), \quad (4,5), \quad (1,4,5), \quad (1,5,4), \quad (2,4), \quad (2,5),$$
$$(4,5), \quad (2,4,5), \quad (2,5,4), \quad (3,4), \quad (3,5), \quad (3,5), \quad (3,4,5), \quad (3,5,4)$$

Each element fixes at least two of 1, 2, and 3.

What is the largest set of permutations in a **permutation group** so that any two are intersecting?

Simple Facts about Intersecting Sets in a Group

For a group G, let $\sigma, \pi, g \in G$:

- Two permutations σ **and** π **intersect if and only if** $\pi^{-1}\sigma$ **has a fixed point.**
  - $\sigma = (1,2,4,3,5)$ and $\pi = (1,4,5)(2,3) \quad \Rightarrow \quad$ intersect since then both map 5 to 1
  - $\sigma^{-1}\pi = (1,2,4,3,5)^{-1}\,(1,4,5)(2,3) = (1,2,4,3) \quad \Rightarrow \quad$ fixes 5.
- A permutation is a **derangement** if it fixes no points.

$$(1,2,3,4,5,6) \quad \text{and} \quad (1,2)(3,4)(5,6) \quad \text{in } \mathrm{Sym}(6)$$

- Permutations σ and π are **intersecting** if and only if $\pi^{-1}\sigma$ is **not a derangement**.
- σ and π intersect if and only if $g\sigma$ and $g\pi$ intersect

$$(g\pi)^{-1}(g\sigma) = \pi^{-1}g^{-1}g\sigma = \pi^{-1}\sigma.$$

- H is an intersecting set if and only if gH is an intersecting set.

We can always assume that the **identity** is in an intersecting set,
and every other element in the set has a fixed point.

Theorem (Deza and Frankl - 1977)

The size of the largest intersecting set in $\mathrm{Sym}(n)$ *is* $(n-1)!$.

First, the stabilizer of any point is an intersecting set of this size.

Next, consider the subgroup $H = \langle (1,2,3,\ldots,n) \rangle$.

1. No pair of elements in H (or in gH) intersect.
2. Partition $\mathrm{Sym}(n)$ by the cosets of H
3. Any intersecting set will contain at most one element from each coset.
4. Any intersecting set will contain at most $\frac{n!}{n} = (n-1)!$ elements. □

Example (for $\mathrm{Sym}(4)$)

H:	()	(1,3)(2,4)	(1,4,3,2)	(1,2,3,4)
x_1H:	(3,4)	(1,4,2,3)	(1,3,2)	(1,2,4)
x_2H:	(2,3)	(1,2,4,3)	(1,4,2)	(1,3,4)
x_3H:	(2,3,4)	(1,4,3)	(1,2)	(1,3,2,4)
x_4H:	(2,4,3)	(1,2,3)	(1,3,4,2)	(1,4)
x_5H:	(2,4)	(1,3)	(1,2)(3,4)	(1,4)(2,3)

Canonical Intersecting Sets

The stabilizer of a point in a group G is an intersecting set

$$G_i = \{\sigma \in G \,|\, i^\sigma = i\}.$$

Any coset of a stabilizer of a point is an intersecting set

$$G_{i,j} = \{\sigma \in G \,|\, i^\sigma = j\}.$$

These are called the **canonical intersecting sets**.

Lemma

Any transitive group G with degree n has an intersecting set of size $\frac{|G|}{n}$.

Question

Are the canonical intersecting sets the largest intersecting sets?

Question

How much bigger than a canonical intersecting sets can an intersecting set be?

Intersection Density

Let $G \leq \mathrm{Sym}(n)$ be any finite **transitive** group.
Only going to consider transitive groups.

1. For $\mathcal{F} \subseteq G$ intersecting, define the **intersection density of** $\mathcal{F}$ to be
$$\rho(\mathcal{F}) = |\mathcal{F}| \left(\frac{|G|}{n} \right)^{-1} = \frac{|\mathcal{F}|}{|G_x|}.$$
2. The intersection density of the **stabilizer** of a point is 1.
3. The **intersection density of the group** G is
$$\rho(G) := \max\{\rho(\mathcal{F}) \,|\, \mathcal{F} \subseteq G \text{ is intersecting}\}.$$

   (This was defined by Li, Song and Pantangi in 2020.)

Observation

The intersection density of any transitive permutation group is at least 1.

Erdős-Ko-Rado Property

Definition

A group has intersection density 1 is said to have the **Erdős-Ko-Rado Property**.

If a group has intersection density 1 then the stabilizer of a point is the largest intersecting set.

Theorem (Erdős-Ko-Rado Theorem - 1961)

Let $\mathcal{A}$ be an intersecting k-set system on an n-set. If $n > 2k$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

The largest intersecting set is the collection of all sets that contain a common point.

Theorem

The group $\mathrm{Sym}(n)$ has the Erdős-Ko-Rado property.

Group Actions

The intersection density is a property of the **group action**, not the group.

1. If G is a transitive group on Ω then this action is equivalent to the **action of G on the cosets** G/H where H is the stabilizer of a point $\omega \in \Omega$.
2. If $\sigma \in G$ fixes a point in its action on G/H, then there is an x with
$$\sigma(xH) = xH, \qquad \text{which implies} \qquad x^{-1}\sigma x \in H.$$
3. We are looking for a set $\mathcal{F}$ so that for any $\sigma, \pi \in \mathcal{F}$ we have $\pi^{-1}\sigma$ is **conjugate** to an element of H.

Example of a Group Action

Example

Consider $\mathrm{Alt}(4)$, acting on **pairs** from $\{1,2,3,4\}$:

- This can be considered as a subgroup of $\mathrm{Sym}(6)$.
- The stabilizer of the pair $\{1,2\}$ is $H = \{(), (1,2)(3,4)\}$.
  This is the canonical intersecting set.
- The set of element in $\mathrm{Alt}(4)$ are conjugate to the elements in H are:

Permutations conjugate to the elements in H	Pairs fixed by the permutation
()	$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$
(1,2)(3,4)	$\{1,2\}, \{3,4\}$
(1,3)(2,4)	$\{1,3\}, \{2,4\}$
(1,4)(2,3)	$\{1,4\}, \{2,3\}$

- $\{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is an intersecting set (actually a subgroup).

Lemma

The group $\mathrm{Alt}(4)$ acting of the pairs from $\{1,2,3,4\}$ does not have the Erdős-Ko-Rado property. Its intersection density is at least 2.

Lemma

A **subgroup** H *is an intersecting set if and only if no element of* H *is a derangement.*

$\rightarrow$ Every element in H intersects with the identity, so is not a derangement.

$\leftarrow$ Assume no element of H is a derangement. For any $\sigma, \pi \in H$, then $\pi^{-1}\sigma \in H$ is not a derangement, so σ and π are intersecting. □

This is an easy way to look for intersecting sets, just check for subgroups with no derangements.

Question

What are the largest intersecting subgroups in a group?

See Mohammad Bardestani, Keivan Mallahi-Karai "On the Erdős-Ko-Rado property for finite Groups"

Simple Bounds

Lemma

If a degree n transitive group G has a subgroup H in which all the elements of H are derangements, then the intersection density of G is at most $\frac{n}{|H|}$.

1. The right cosets of H partition the elements of G,
2. an intersecting set can have at most one element from each coset.
3. The intersection density is bounded by

$$\frac{\frac{|G|}{|H|}}{\frac{|G|}{n}} = \frac{n}{|H|}.$$

Lemma

The intersection density of $\mathrm{Alt}(4)$ acting of the pairs from $\{1,2,3,4\}$ is 2.

Consider the subgroup $H = \{(), (1,2,3), (1,3,2)\}$.

Another Action

Example (Hujdurović, Kovács, Kutnar, Marušič)

What is the intersection density of $\mathrm{Sym}(n)$ with its action on $\mathrm{Sym}(n)/\mathbb{Z}_3$?

The degree is $\frac{n!}{3}$.

1. Find the largest set $\mathcal{F}$ of permutations in $\mathrm{Sym}(n)$ so that for any $x, y \in \mathcal{F}$

$$xy^{-1} \text{is a 3-cycle.}$$

2. Assume identity is in $\mathcal{F}$; so all other elements are 3-cycles.
3. Consider the elements

$$(),\ (1,2,3),\ (1,2,4),\ \dots,\ (1,2,n)$$

4. This set has size $1 + (n-2) = n - 1$.
5. A cycle that intersects with $(1,2,3)$ must be of the form: $\{(1,2,x),(1,x,3),(x,2,3)\}$.
   So this set is the largest possible
6. The intersection density is

$$\frac{n-1}{3}$$

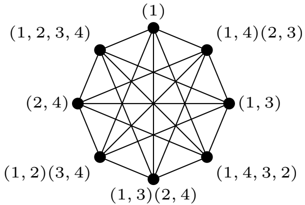
So no absolute bound on intersection density.

Definition

For any permutation group G we can define a **derangement graph**, Γ_G.

- The vertices are the elements of G.
- Vertices $\sigma, \pi \in G$ are adjacent if and only if $\pi^{-1}\sigma$ is a derangement.

(So permutations are adjacent if they are **not** intersecting.)



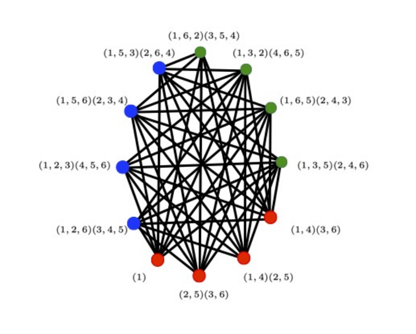
The graph $\Gamma_{D(4)}$.

An intersecting set in G is a coclique (independent set / stable) in Γ_G.

Example, $\mathrm{Alt}(4)$ on Pairs

Example

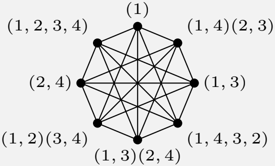
This is the derangement graph for $\mathrm{Alt}(4)$ on the pairs from $\{1, \ldots, 4\}$:



It is the complete multipartite graph $K_{4,4,4}$, and the maximum coclique has size 4.

It is easy to see that the intersection density of this group with this action is 2.

Properties of Derangement Graph



The graph $\Gamma_{D(4)}$.

- This graph is **regular**, all the vertices have the same number of neighbours. The number of neighbours (**degree**) is the number of derangements.
- The derangement graph is the **Cayley graph** $\mathrm{Cay}(G, \mathrm{der}(G))$ where $\mathrm{der}(G)$ is the set of derangements of G.
  The vertices are elements of G and x, y are adjacent if $xy^{-1} \in \mathrm{der}(G)$.
- G is a subgroup of the automorphism group of Γ_G.
- This graph is **vertex transitive**: the automorphism group acts transitively on the vertices (all the vertices are the same).

Easy Derangement Graphs

Complete Graph

- If the derangement graph is a complete graph, then every element in the group is a derangement.
- This happens if and only the group is regular.
  For each i and j there is a unique g such that $i^g = j$.

Union of Complete Graphs

- The derangement graph is a union of complete graphs if and only if G is a transitive Frobenius group
- A maximum coclique has one vertex from each for component.

Complete Multipartite Graph

- A maximum coclique is the set of vertices in a component.

Complete multipartite Derangement Graphs

Question

Which groups have a derangement graph that is a complete multipartite graph?

Question

Can every complete multipartite graph be the derangement graph of some group?

Example

$K_{2,2}$ is not a derangement graph for any group.

1. If $G = C_4$, then
$$G = \{(),\ (1,2,3,4),\ (1,3)(2,4),\ (1,4,3,2)\}$$
so $\Gamma_G = K_4$
2. If $G = C_2 \times C_2$, then
$$G = \{(),\ (1,2),\ (3,4),\ (1,2)(3,4)\}$$
so $\Gamma_G = K_2 \cup K_2$.

Complete multipartite Derangement Graphs

Lemma (M., Spiga, Razafimahatratra)

For every $n = 2k$, with k **odd**, *there is a group G with degree n and Γ_G a* **complete multipartite graph with k parts**.

Set

$$H = \langle (1,2), (3,4), \ldots, (n-1,n) \rangle$$

and

$$G = \mathrm{Alt}(n) \cap \langle H, (1,3,\ldots,n-1)(2,4,\ldots,n) \rangle. \quad \square$$

Lemma (Hujdurović, Kutnar, Marušič and Miklavič)

For every $n = 2^a k$, with k odd, there is a group G with degree n and Γ_G a complete multipartite graph with k parts.

1. Let H be a regular group with degree 2^{a-1} (so Γ_H is complete graph).
2. Let K be the group from the previous lemma.
3. $G = H \times K$ has Γ_G a complete multipartite graph with $2^{a-1}k$ parts. $\square$

Definition

Define H_G to be the subgroup generated by the elements of G that fix at least one point, that is,

$$H_G = \langle \bigcup_{w \in \Omega} G_w \rangle = \langle G \setminus \mathrm{der}(G) \rangle.$$

Example

$H = \mathrm{Alt}(4)$ acting on pairs from $\{1, 2, 3, 4\}$.

$$\begin{aligned} H_G &= \langle (), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) \rangle \\ &= \{ (), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) \} \end{aligned}$$

This is a proper subgroup of $\mathrm{Alt}(4)$, and every element fixes a pair.

Here, the non-derangements don't generate any derangements

Complete Multi-partite Derangement graphs

Lemma

Let G be transitive.
Γ_G *is a* **complete multipartite graph** *if and only if* H_G **is a derangement-free, proper subgroup** *of* G*.*

Consider the complement of Γ_G, this is $\mathrm{Cay}(G, C)$, where C is the set of non-identity elements that have a fixed point.

- The connected component of $\mathrm{Cay}(G, C)$ that contains the identity is the group generated by C.
- $H_G = \langle C \rangle$.
- $\mathrm{Cay}(G, C)$ is the union of complete graphs if and only if H_G is a derangement-free proper subgroup.
- So Γ_G is complete multipartite graph if and only if H_G is a proper derangement-free subgroup. $\square$

In this case Γ_G is a complete multi-partite graph with $[G : H_G]$ parts and $\rho(G) = \frac{n}{[G:H_G]}$.

Simple Bounds

Lemma (Clique-Coclique bound)

If X is a vertex-transitive graph

$$\alpha(X)\,\omega(X) \leq |V(X)|$$

$\alpha(X)$ is the size of the largest coclique, $\omega(X)$ is the size of the largest clique.

This is really Frankl and Deza's proof that the intersection density of $\mathrm{Sym}(n)$ is 1.

Lemma

If G is degree n transitive group then the intersection density no more than $\frac{n}{2}$.

A transitive group has a derangement, so Γ_G has an edge, which is a clique of size 2.

Example (Really a Silly Example)

The intersection density of $\mathrm{Sym}(2)$ is $1 = \frac{2}{2} = \frac{n}{2}$.

Question

Are there other degree n groups that have intersection density $\frac{n}{2}$?

Bipartite Derangement Graph

Theorem (M., Razafimahatratra, Spiga)

Provided that $n > 2$, the derangement graph for a transitive group with degree n is not bipartite.

Assume Γ_G is bipartite,

1. The part of Γ_G that contains the identity is a normal subgroup, H.
2. This H fixes the bipartition.
3. The subgroup H has no derangements, so it can't be transitive.
4. H has 2 orbits in the group action.
5. If ω and ω' are from different orbits, we can show

$$H = \bigcup_{h \in H} H_\omega^h \cup \bigcup_{h \in H} H_{\omega'}^h.$$

6. This means that H has **normal covering number two**.
7. By examining the characterisation of groups with normal cover two (by M. Garonzi, A. Lucchini), such H exists only if $n = 2$. □

Theorem (M., Razafimahatratra, Spiga)

Let G be a transitive permutation group with degree $n \geq 3$, then the derangement graph of G contains a triangle.

Corollary

For any group G with degree $n \geq 3$, we have $\rho(G) \leq \frac{n}{3}$.

This follows from the previous result and the clique-coclique bound,

$$3\,\alpha(\Gamma) \leq \omega(\Gamma)\,\alpha(\Gamma) \leq |V(\Gamma)|. \quad \square$$

Question

Are there many of groups with $\rho(G) = \frac{n}{3}$?

Example of a Derangement Graph

Example (Razafimahatratra)

Let G:=TransitiveGroup(18, 142).

- This is an imprimitive with size 324 (it has a system with three blocks of size six.)
- The derangement graph for this graph is a complete tripartite graph.
- $$\rho(G) = \frac{108}{\frac{324}{18}} = 6 = \frac{n}{3}$$

Using (n, d) in the database of `TransitiveGroup` in `gap` we found 4 groups

1. $(6, 4)$ having degree 6 and order 12,
2. $(18, 142)$ having degree 18 and order 324,
3. $(30, 126)$ having degree 30 and order 600,
4. $(30, 233)$ having degree 30 and order $1\,200$.

All the examples of degree n groups with intersection density $\frac{n}{3}$ are complete tripartite.

It is easy to check if the derangement graph is complete multipartite by the eigenvalues!

Eigenvalues of Cayley Graphs

The derangement graph is a *normal* Cayley graph

$$\Gamma_G = \mathrm{Cay}(G, \mathrm{der}(G)).$$

1. The *connection set* is the set of derangements and is **closed under conjugation**.
2. It is actually a union of conjugacy classes.

Theorem (Babai & Diaconis and Shahshahani)

If $Cay(G, C)$ is a normal Cayley graph, then its eigenvalues are

$$\frac{1}{\chi(1)} \sum_{\sigma \in C} \chi(\sigma)$$

where χ is an irreducible character of G.

For χ an irreducible character of G, the eigenvalue of Γ_G belonging to χ is

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{\sigma \,\text{derangement}} \chi(\sigma) = \frac{1}{\chi(1)} \sum_{\substack{C \,\text{conjugacy class of derangements} \\ c \in C}} |C| \, \chi(c)$$

Eigenvalues of Cayley Graphs

Example

Let $\mathbf{1}$ be the trivial character for G, then

$$\lambda_{\mathbf{1}} = \frac{1}{\mathbf{1}(1)} \sum_{g \in \mathrm{der}(G)} \mathbf{1}(g) = |\,\mathrm{der}(G)| = d.$$

This is the degree of the derangement graph.

Example

Let $\psi(g) = \mathrm{fix}(g) - 1$, this is an irreducible character if G is 2-transitive

$$\lambda_{\psi} = \frac{1}{\psi(1)} \sum_{g \in \mathrm{der}(G)} \psi(g) = \frac{-d}{n-1}.$$

Hoffman-Delsarte Ratio Bound

Ratio Bound

If X is a vertex-transitive graph, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where d is the degree and τ is the least eigenvalue of the adjacency matrix of X.

Example

Let G be a 2-transitive graph. If $\frac{-d}{n-1}$ is the least eigenvalue of the derangement graph of G, then

$$\alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{d}{\frac{-d}{n-1}}} = \frac{|G|}{n}$$

and the group has intersection density 1.

Question

For which 2-transitive groups is λ_ψ the least eigenvalue of G?

Lots, but not all 2-transitive groups have this property!

Weighted Adjacency matrix

A *weighted* adjacency matrix for a graph X is a

1. $|V(X)| \times |V(X)|$
2. symmetric matrix with
3. the (i,j)-entry non-zero only if vertices i and j are adjacent in X.
4. Put a weight on the edges, can weight them with 0.

Ratio Bound for Weighted Adjacency Matrices

If A is the weighted adjacency matrix for a vertex-transitive graph Γ then

$$\alpha(\Gamma) \leq \frac{|V(\Gamma)|}{1 - \frac{d}{\tau}}$$

d is the row sum and τ is the least eigenvalue for a weighted adjacency matrix of Γ.

Derangement Graphs

A derangement graph Γ_G is the union of Cayley graphs—one Cayley graph for each conjugacy class of derangements.

$$\Gamma_G = Cay(G, Der(G)) = \bigcup_C Cay(G, C)$$

where the union is taken over the conjugacy classes of derangements.

The adjacency matrix of the derangement graph is

$$A(\Gamma_G) = \sum_{\substack{C \\ \text{conjugacy class of derangements}}} A(Cay(G, C)).$$

We can form a **weighted adjacency matrix** by weighting the conjugacy classes

$$A(\Gamma_G) = \sum_{\substack{C \\ \text{conjugacy class of derangements}}} w_C \; A(Cay(G, C)),$$

then the eigenvalues are (where $c \in C$)

$$\lambda_\chi = \sum_{\substack{C \\ \text{conjugacy class of derangements}}} w_C \; |C|\chi(c).$$

Set this up as a linear programming problem:

- Put weights on the conjugacy classes of derangements.
- Maximize the eigenvalue from the trivial character,
- while keeping all other eigenvalues above -1.

Focus on the Permutation character:

1. The permutation character minus the trivial character is

$$\psi(g) = \mathrm{fix}(g) - 1.$$

2. Set the weightings on the conjugacy classes so that all irreducible characters in the decomposition ψ give the eigenvalue of -1.

Cases where this method works to show a group has EKR property

1. Symmetric group natural action
2. Alternating group natural action
3. Symmetric group on ordered t-tuple (Ellis, Freidgut and Pilpel)
4. Symmetric group on t-sets (Ellis)
5. $\mathrm{PGL}(n, q)$ (Spiga)
6. $GL(n, q)$ groups (Schmidt and Ernst)

See: Alena Ernst "Erdős-Ko-Rado theorems for finite general linear groups" Saturday 9:50 in RAM - Oval

$\Rightarrow$

t-Intersecting Permutations

Theorem (Ellis, Friedgut, Pilpel 2010)

For n sufficiently large, $\mathrm{Sym}(n)$ acting on the cosets of $\mathrm{Sym}(n-t)$ has intersection density 1.

This is $\mathrm{Sym}(n)$ acting on ordered t-sets.

Theorem (Ellis, 2011)

For n sufficiently large, $\mathrm{Sym}(n)$ acting on the cosets of $\mathrm{Sym}(t) \times \mathrm{Sym}(n-t)$ has intersection density 1.

This is $\mathrm{Sym}(n)$ acting on unordered t-sets.

Conjectures

Pointwise action:

Conjecture

If $n \geq 2t+1$ then $\mathrm{Sym}(n)$ acting on the cosets of $\mathrm{Sym}(n-t)$ has intersection density 1.

True for $t = 2$ M. and Razafimahatratra.

Setwise action:

Conjecture

If $n \geq t$ then $\mathrm{Sym}(n)$ acting on the cosets of $\mathrm{Sym}(n-t) \times \mathrm{Sym}(n-t)$ has intersection density 1.

1. True for $t = 2$ M. and Razafimahatratra
2. True for $t = 3$ Behajaina, Maleki, Rasoamanana and Razafimahatratra.
3. True for $t = 4, 5$ Behajaina, Maleki, and Razafimahatratra.

Theorem (M., Spiga, Tiep)

All 2-transitive groups have intersection density 1.

First we used two reductions:

1. if a group has a sharply transitive subgroup (a subgroup with all elements a derangement) then it has intersection density 1.
2. if G has a transitive subgroup H with intersection density 1, then G has intersection density 1.

We only needed to look at minimal transitive subgroups of almost simple type.

We can go through these all groups and apply the ratio bound

Characterisations/Open Problems

Theorem (M., Sin)

Let G be a 2-transitive group.
The characteristic vector of any maximum intersecting set is a linear combination of the characteristic vectors of the canonical intersecting sets.

This can be used to characterise all the maximum intersecting set

Question

What are all the largest intersecting sets in the 2-transitive groups?

Question

When are all the largest intersecting set in a 2-transitive group either a subgroup of the coset of a subgroup?

Open Questions

Lemma

If G is a group with degree n, if the intersection density is $n/2$, then $n \leq 2$.

Is there a similar result, like:

Question

If G is a group with degree n, if the intersection density is $n/3$, then is n bounded by something?

Question

Start with a vertex-transitive graph.
What is the largest intersection density of all the transitive subgroups of the automorphism group?

See: A. Sarobidy Razafimahatratra "Intersection density of vertextransitive graphs" Friday 18:25 in KOM-1

Open Questions

Question

For a given degree, what are the intersection densities of all the transitive subgroups with the degree?

Lemma

Let p be a prime. If G is a transitive group with degree p, then the intersection density is of G is 1.

Theorem (Hujdurović, Kovács, Kutnar, Marušič)

If G is a transitive group with degree pq for p and q odd primes, then the intersection density is of G either 1 or 2.