Directed max-cut and some generalizations

Anders Yeo

yeo@imada.sdu.dk
Department of Mathematics and Computer Science
University of southern Denmark
Campusvej 55, 5230 Odense M, Denmark

Joint work with: Jiangdong Ai, Argyrios Deligkas, Eduard Eiben, Stefanie Gerke, Gregory Gutin, Philip R. Neary and Yacong Zhou

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We will consider the directed max-cut problem and some of its generalizations.

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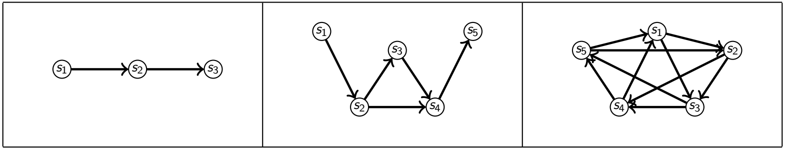

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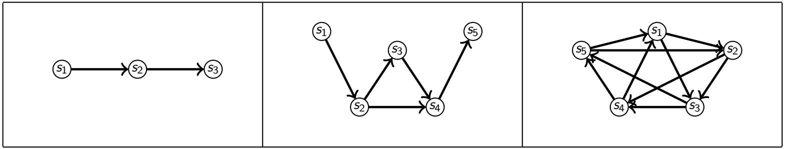

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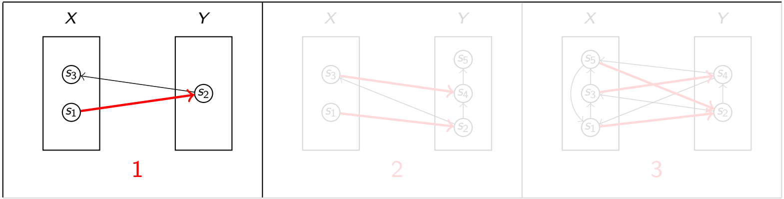

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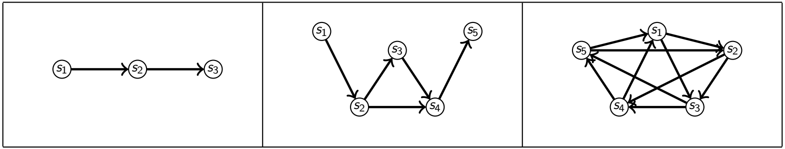

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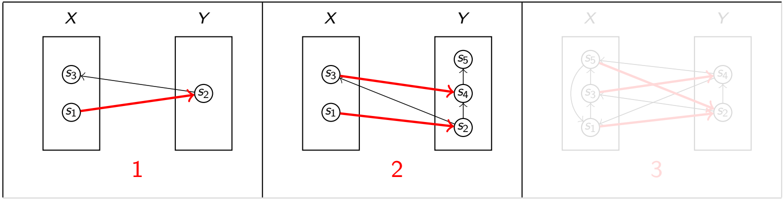

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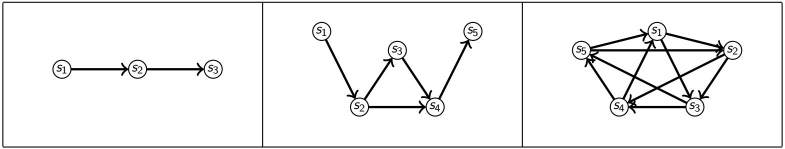

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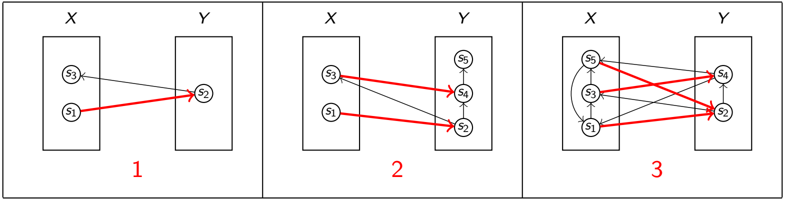

Basic bounds

Let $mac(D)$ denote the maximum number of arcs in a (X, Y)-cut in a digraph D and let $a_D(X, Y)$ denote the number of (X, Y)-arcs in D.

Analogously, let $mac(G)$ denote the maximum number of edges in a (X, Y)-cut in a (undirected) graph G.

Theorem 1: $mac(D) \geq \frac{|A(D)|}{4}$ for all digraphs D.
$mac(G) \geq \frac{|A(G)|}{2}$ for all graphs G.

Proof: place every vertex randomely in X or Y with equal probability (50%).

The above bounds are the average number of arcs/edges in the cut. QED

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Theorem 1 can be improved to the following.

Theorem 2: $mac(D) \geq \frac{|A(D)|}{4} + \frac{|A(D)|}{4n}$ for all digraphs D of order n.

Proof: Randomely place $\lfloor \frac{n}{2} \rfloor$ vertices in X and the remaining vertices in Y.

If n is odd then $P(x \in X \ \& \ y \in Y) = \frac{\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil}{n(n-1)} = \frac{1}{4} + \frac{n-1}{4n(n-1)}$

So the average number of arcs in the cut is $\frac{|A(D)|}{4} + \frac{|A(D)|}{4n}$.

When n is even we get that the average is $\frac{|A(D)|}{4} + \frac{|A(D)|}{4(n-1)} \geq \frac{|A(D)|}{4} + \frac{|A(D)|}{4n}$. QED

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Regular digraphs

Question: If D is a eulerian digraph (ie $d^+(x) = d^-(x)$ for all x), what is $mac(D)$ (in terms of $mac(G)$)?

Answer: $mac(D) = \frac{mac(UG(D))}{2}$. Why?

Let $G = UG(D)$ and let (X, Y) be any cut in G.

As $d^+(x) = d^-(x)$ for all $x \in V(D)$ we have $a_D(X, Y) = a_D(Y, X)$ (as any eulerian tour enters and leaves X equally many times in D).

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Regular tournament

A tournament is an orientation of a complete graph.

Theorem 6: If T is a regular tournament of order n then $mac(T) = \frac{1}{2} \cdot \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n^2}{8} \rfloor$.

Proof: As T is eulerian we note that $mac(T) = \frac{mac(K_n)}{2} = \frac{1}{2} \cdot \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor = \frac{1}{2} \cdot \lfloor \frac{n^2}{4} \rfloor = \lfloor \frac{n^2}{8} \rfloor$. QED

For a regular tournament T of order n and size m we have $m = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$, so $mac(T) = \lfloor \frac{n^2}{8} \rfloor = \lfloor \frac{m}{4} + \frac{n}{8} \rfloor$.

So, the maximum cut contains slightly more than a quarter of the arcs ($mac(T) \approx \frac{m}{4} + \frac{1+\sqrt{1+8m}}{16} \approx \frac{m}{4} + \sqrt{\frac{m}{32}}$).

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We will consider the weighted version

We give a weight for each arc and we want to find a cut (X, Y) where the sum of the weights of all (X, Y)-arcs is maximum.

Let $w^+(x)$ denote the sum of the weight on the arcs leaving x and let $w^-(x)$ denote the sum of the weight on the arcs entering x.

$w^+(s_2) = 3$ and $w^-(s_2) = 1$.

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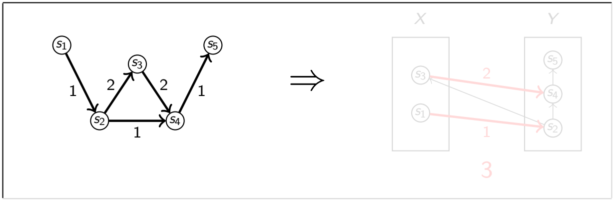

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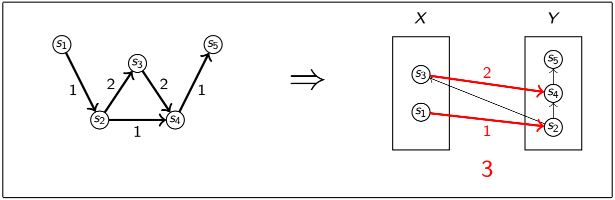

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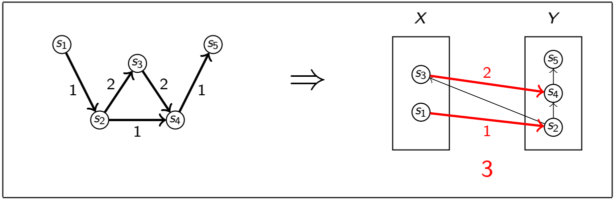

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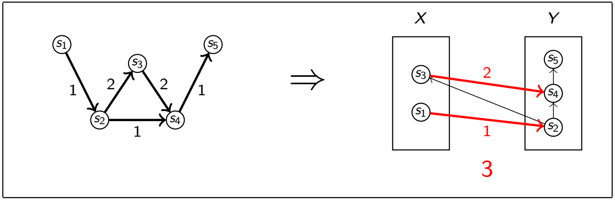

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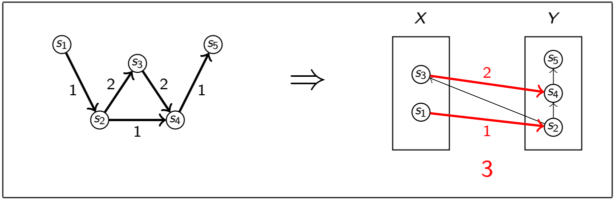

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Let $\theta(D) = \frac{\sum_{x \in V(D)} \max\{0, w^+(x) - w^-(x)\}}{w(D)}$.

What is $\theta(D)$ of the shown digraph?

$\theta(D) = \frac{(1-0)+(3-1)}{7} = \frac{3}{7} \approx 0.43$

If D is weighted-eulerian ($w^+(x) = w^-(x)$ for all x) then $\theta(D) = 0$.

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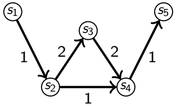

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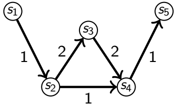

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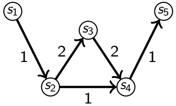

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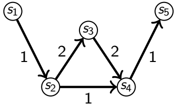

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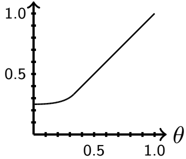

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So, if $\theta(D) > 0$ we can improve the bound $mac(D) \geq w(D)/4$.

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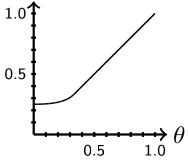

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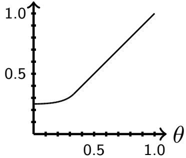

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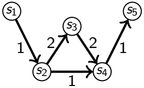

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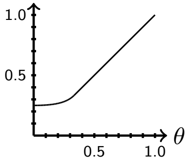

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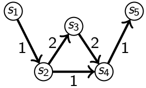

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To show the bound is tight we let D_k be a digraph consisting of two vertex disjoint regular tournament, A_k and B_k, of order k and arc-weights 1.

We then add all arcs from A_k to B_k with weight $Q = \frac{\theta(1-1/k)}{1-\theta}$.

One can then show that $\theta(D_k) = \theta$ and $mac(D_k)$ approaches $l(\theta) \cdot w(D)$, when k goes to infinity.

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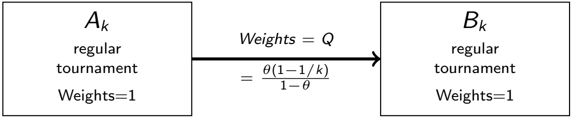

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Acyclic digraphs

For (unweighted) acyclic digraphs Alon, Bollobas, Gyarfas, Lehel and Scott proved the following

Theorem 8 (Alon et al): There exists a constant k_1^s, such that for every integer $m \geq 1$ there exists an acyclic digraph D_m^s with m arcs and $mac(D_m^s) \leq \frac{m}{4} + k_1^s m^{0.8}$.

Theorem 9 (Alon et al): There exists a constant k_2^s, such that $mac(D) \geq \frac{m}{4} + k_2^s m^{0.6}$ for all acyclic digraphs D of size m.

Recall that for general digraphs the regular tournament, T_n, of order n and size m, satisfies $mac(T_n) \approx \frac{m}{4} + \sqrt{\frac{1}{32}} \times m^{0.5}$

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We in fact generalize to multi-digraphs and arc-weighted digraphs.

Theorem 10, [1]: There exists a constant k_1, such that for every integer $m \geq 1$ there exists an acyclic multi-digraph D_m with m arcs and $mac(D_m) \leq \frac{m}{4} + k_1 m^{0.75}$.

Theorem 11, [1]: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all acyclic arc-weighted digraphs D with $w \geq 1$.

Theorem 10 and 11 hold for both multi-digraphs and arc-weighted digraphs ($w \geq 1$).

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We first outline the proof of Theorem 10.

Let $V(D) = \{v_1, v_2, \ldots, v_n\}$ and add an acyclic tournament on $I_i = (v_i, v_{i+1}, \ldots, v_{i+q-1})$ where all arcs go "forward" in the order of I_i and all indices are taken modulo n.

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Theorem 10

The result we call D_m^*, which is a regular multi-digraph.

We now delete all "backward" arcs and call the result D_m.

As $A(D_m^*)$ can be partitioned into n tournaments on q vertices we note that $mac(UG(D_m^*)) \leq n \cdot mac(K_q) = n \cdot \lfloor \frac{q^2}{4} \rfloor \leq \frac{nq^2}{4}$.

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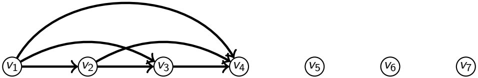

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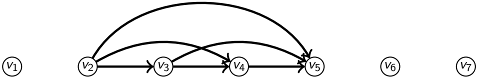

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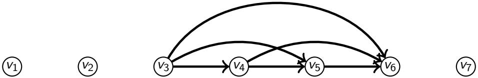

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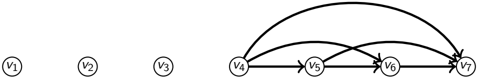

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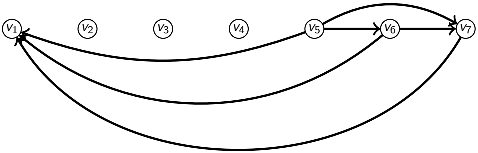

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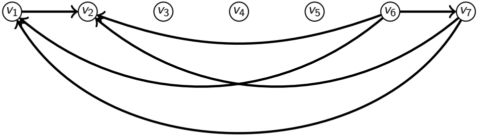

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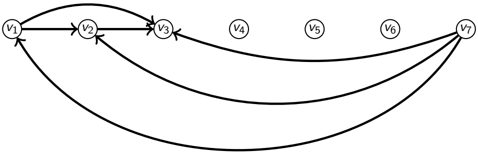

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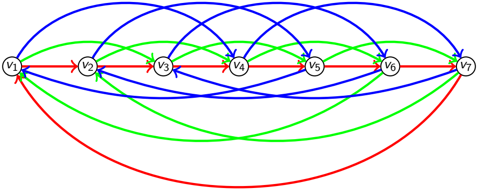

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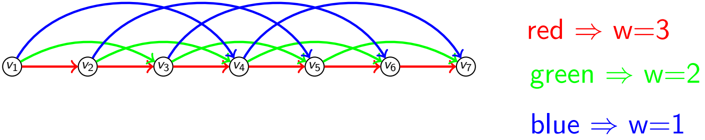

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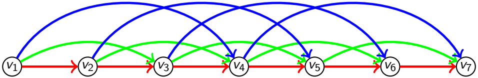

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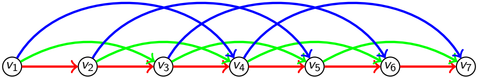

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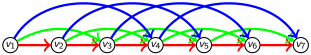
Theorem 10 (The boring computations)

Example

$n = 7$ and $q = 4$:



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One can then extend this to all values of m....

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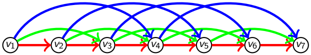

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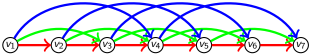

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Theorem 11

Recall Theorem 11.

Theorem 11, [1]: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all arc-weighted acyclic digraphs D $(w \geq 1)$.

In order to prove this we need a result on arc-weighted acyclic digraphs with maximum path containing ν vertices.

Let c_ν be the largest number such that $mac(D) \geq c_\nu \times w(D)$ for all arc-weighted acyclic digraphs D with maximum path order at most ν.

Theorem 12, [1]: $c_\nu \geq \frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}$.

Proving Theorem 12 is the main part in proving Theorem 11. We will not give the proof of Theorem 12, but just note that the approach is completely different then for Theorem 10 (the Alon result).

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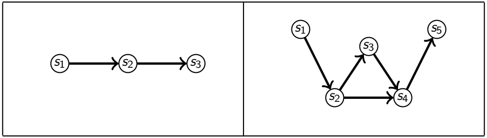

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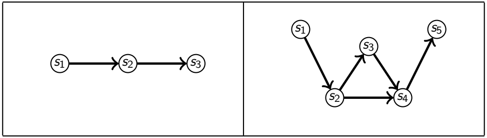

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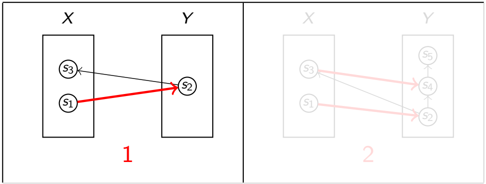

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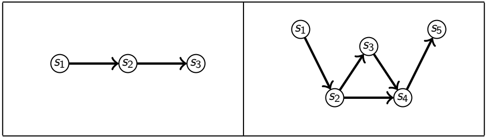

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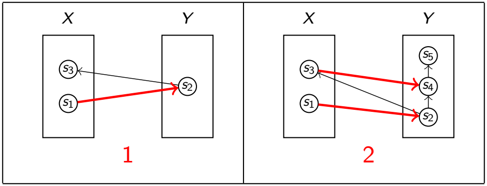

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Proof: Let D be a arc-weighted acyclic digraphs D.

Let $P = p_1 p_2 p_3 \ldots p_n$ be a longest path in D.

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Theorem 11, Case 1 proof

Case 1: $w(P) \leq w(D)^{0.6}$.

Theorem 12:

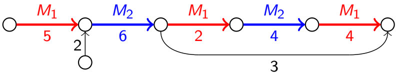
$c_\nu \geq \frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}$.

As all weights are at least one, we have $|A(P)| \leq w(P) \leq w(D)^{0.6}$. So Theorem 12 implies,

$$\begin{aligned} mac(D) &\geq \left(\frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times |A(P)|^{2/3}}\right) w(D) \\ &\geq \frac{w(D)}{4} + \frac{w(D)}{8 \times 3^{2/3} \times w(D)^{0.4}} \\ &\geq \frac{w(D)}{4} + k_2 \cdot w(D)^{0.6} \end{aligned}$$

Theorem 11, Case 2 proof

Case 2: $w(P) \geq w(D)^{0.6}$.



Let M_1 and M_2 be two matchings in $A(P)$ such that $A(M_1) \cup A(M_2) = A(P)$.

W.l.o.g assume that $w(M_1) \geq w(M_2)$, and for each arc, uv, in M_1 assign u to X and v to Y with probability $1/2$ and assign u to Y and v to X with probability $1/2$. Any vertex not in $V(M_1)$ gets assigned to X or Y with probability $1/2$.

The average weight of the cut (X, Y) is the following.

$$\frac{w(D)}{4} + \frac{w(M_1)}{4} \geq \frac{w(D)}{4} + \frac{w(P)/2}{4} \geq \frac{w(D)}{4} + \frac{w(D)^{0.6}}{8}$$

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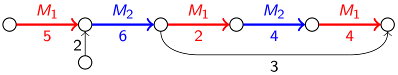

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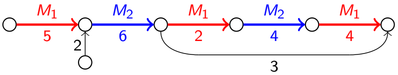

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Open problem

Theorem 10, [1]: There exists a constant k_1, such that for every integer $m \geq 1$ there exists an acyclic multi-digraph D_m with m arcs and $mac(D_m) \leq \frac{m}{4} + k_1 m^{0.75}$.

Theorem 11, [1]: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all arc-weighted acyclic digraphs D $(w \geq 1)$.

Open Problem: Close the gap between 0.6 and 0.75 for arc-weighted acyclic digraphs D.

Final open problem

For simple digraphs the following holds.

Theorem 8 (Alon et al): There exists a constant k_1^s, such that for every integer $m \geq 1$ there exists an acyclic digraph D_m^s with m arcs and $mac(D_m^s) \leq \frac{m}{4} + k_1^s m^{0.8}$.

Theorem 9 (Alon et al): There exists a constant k_2^s, such that $mac(D) \geq \frac{m}{4} + k_2^s m^{0.6}$ for all acyclic digraphs D of size m.

Open Problem: Close the gap between 0.6 and 0.8 for simple acyclic digraphs D.

End of first part of the talk

This completes the first part of the talk, which was based on the paper

[1] Jiangdong Ai, Stefanie Gerke, Gregory Gutin, Anders Yeo and Yacong Zhou. *Bounds on Maximum Weight Directed Cut.* Submitted.

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Generalization of max-cut in digraphs

Let D be a digraph, such that for each arc $a \in A(D)$ we are given values $xx(a)$, $xy(a)$, $yx(a)$ and $yy(a)$. We want to find a partition (X, Y) of $V(D)$ that maximizes $\sum_{a \in A(D)} val(a)$, where

$$val(uv) = \begin{cases} xx(uv) & \text{if } u, v \in X \\ xy(uv) & \text{if } u \in X \text{ and } v \in Y \\ yx(uv) & \text{if } u \in Y \text{ and } v \in X \\ yy(uv) & \text{if } u, v \in Y \end{cases}$$

We denote the values $(xx(a), xy(a), yx(a), yy(a))$ by

$$M(uv) = \begin{bmatrix} xx(uv) & xy(uv) \\ yx(uv) & yy(uv) \end{bmatrix}$$

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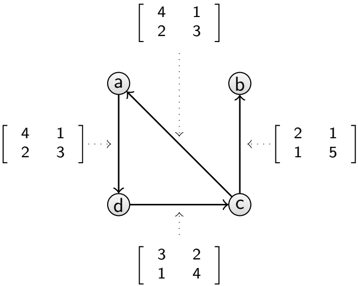

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The optimal partition is $(\{a, d\}, \{b, c\})$ with value 13.

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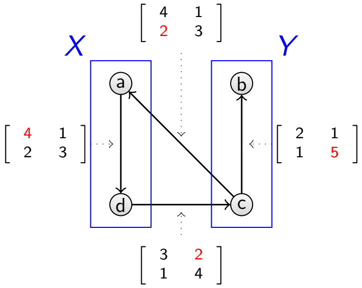

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Generalization of max-cut in dgraphs

In order to obtain a dichotomy, we will let $\mathcal{F}$ denote the list of matrices that are allowed.

We assume that if a matrix $M \in \mathcal{F}$ is allowed to be used then every multiple of M is also allowed to be used.

Example: The directed max-cut problem (we count the number of (X, Y)-arcs) can be reduced to the case when $\mathcal{F} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.

So, if $\mathcal{F} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ then the problem is NP-hard.

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Dichotomy

We are looking at the problem $MWDP(\mathcal{F})$ (Maximum Weighted Digraph Partition).

We are given a digraph, D, and functions $f : A(D) \rightarrow \mathcal{F}$ and $c : A(D) \rightarrow \mathbb{R}^+$, such that the matrix $c(a) \cdot f(a)$ is used on arc a.

Given $\mathcal{F}$ we define the following 3 properties.

(a): $m_{11} + m_{22} \geq m_{12} + m_{21}$ for all matrices $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in \mathcal{F}$.

(b): $m_{11} \geq \max\{m_{12}, m_{21}, m_{22}\}$ for all matrices $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in \mathcal{F}$.

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Theorem 15, [2]: $MWDP(\mathcal{F})$ is polynomial if Property (a), Property (b) or Property (c) holds and NP-hard otherwise.

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(b): $m_{11} \geq \max\{m_{12}, m_{21}, m_{22}\}$ for all matrices $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in \mathcal{F}$.

(c): $m_{22} \geq \max\{m_{11}, m_{12}, m_{21}\}$ for all matrices $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in \mathcal{F}$.

Theorem 15, [2]: $MWDP(\mathcal{F})$ is polynomial if Property (a), Property (b) or Property (c) holds and NP-hard otherwise.

Proofs

We will not go through the proof of Theorem 15, but instead give some applications.

However, note that if Property (b) or Property (c) hold then the problem is trivially polynomial (by letting $X = V(D)$ or $Y = V(D)$).

If Property (a) holds then we can reduce the problem to finding a (s, t)-minimum cut in an auxilary digraph.

The NP-hardness results require looking at a number of cases and using different techniques for each.

One can also show that the same dichotomy holds even if we require D to be symmetric (or require it to be oriented).

Depending on time, we now give some applications of the dichotomy...

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Application 1, Poly-matrix games, from economics

We are given a number of players, which we think of as vertices in a graph, G. Each player has to chose Strategy 1 or Stratergy 2.

An edge $uv \in A(D)$ indicates that there is a pay-off depending on the stratergies players u and v have chosen.

Let $M_u(uv) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ be the matrix associated with edge uv, such that u gets pay-off m_{ij} if and only if player u choses Stratergy i and player v choses Stratergy j.

Analogously, we define $M_v(uv) = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}$ to indicate player v's pay-off.

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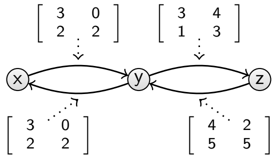
Application 1, Example

Maybe z's payout is twice as important as everyone elses.
So,...



We set $c(xy) = 1$, $c(yx) = 1$, $c(yz) = 1$ and $c(zy) = 2$

The above is an instance of MWDSP($\{R_1, R_2, R_3\}$), where $R_1 = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}$, $R_2 = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$ and $R_3 = \begin{bmatrix} 4 & 2 \\ 5 & 5 \end{bmatrix}$.

MWDSP($\{R_1, R_2, R_3\}$) is a polynomial time solvable problem, as R_1, R_2 and R_3 all satisfy Property (a).

The optimum is for x and y to play strategy 1, and z strategy 2 and the payout is $3 + 3 + 4 + 2 \cdot 5 = 20$.

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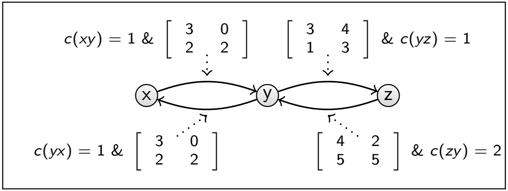

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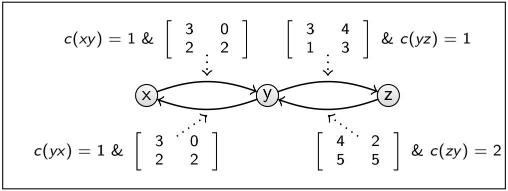

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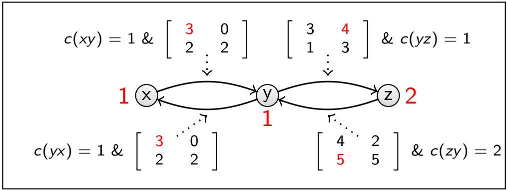

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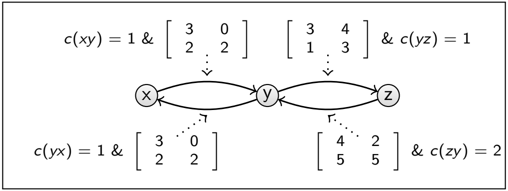

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This problem was originally raised when all matrices have zero's in the off-digaonal ($\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$), which our dichotomy now proves is polynomial.

Our results also indicate why "coordination-games" are easy and "anti-coordination-games" are difficult (in general).

Our results can also be used to determine the complexity of maximizing the potential of the game.

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Application 2, Directed Min (s,t)-cut

Given a digraph, D, with $s, t \in V(D)$, find a (s,t)-partition (X_1, X_2) with the fewest number of arcs from X_1 to X_2.

This is equivalent to finding the largest number of arc-disjoint paths from s to t (by Menger's Theorem).

Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$S = \begin{bmatrix} |A(D)| & 0 \\ 0 & 0 \end{bmatrix}$

$T = \begin{bmatrix} 0 & 0 \\ 0 & |A(D)| \end{bmatrix}$.

Now the maximum value we can obtain is $3|A(D)|$ minus the size of a minimum (s,t)-cut. So by our dichotomy result this is polynomial.

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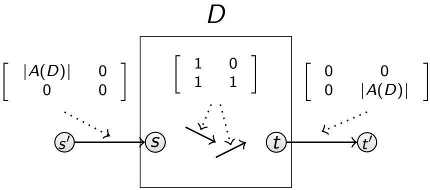

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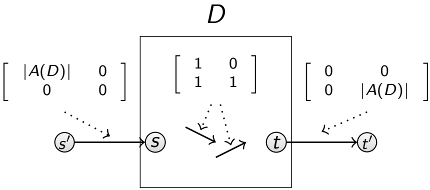

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Application 3, Max Average Degree

Given a graph, G, and an integer k, find a vertex set $X \subseteq V(G)$ such that the induced subgraph $G[X]$ has average degree strictly greater than k.

Let $M_1 = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathcal{F} = \{M_1, M_2\}$.

Let D be any orientation of G after adding a pendent edge to each vertex ($|V(D)| = 2|V(G)|$).

Associate M_1 to each pendent arc and M_2 to all other arcs of D.

This gives us an instance of $MWDP(\mathcal{F})$ and let (X, Y) be an optimal solution. The value of this is the following ($x = |X \cap V(G)|$ and $y = |Y \cap V(G)|$).

$s = k \cdot x + 2e(Y, Y) = k|V(D)| - k \cdot y + 2e(Y, Y)$.

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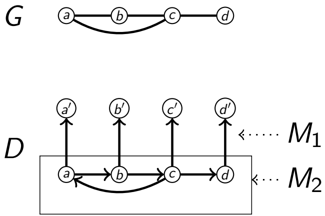

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So, $s > k|V(D)|$ if and only if $2e(Y, Y) > k|Y|$.

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This is equivalent with $k < \frac{2e(Y,Y)}{|Y|} = \frac{\sum_{y \in Y} d_Y(y)}{|Y|} = \text{Avg-deg}(Y)$.

So, there exists a subgraph with average degree greater than k if and only if the solution to $MWDP(\mathcal{F})$ is greater than $k|V(D)|$.

By our dichotomy this implies that the Max-average-degree problem is polynomial.

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Application 4, Max Density

Given a graph, G, find a vertex set $X \subseteq V(G)$ such that the number of edges divided by the number of vertices in the induced subgraph $G[X]$ is maximum possible.

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Application 5, 2-color partition

Given a 2-edge-colored graph, G, find a partition (X_1, X_2) which maximizes the sum of the number of edges in X_1 of color one and the number of edges in X_2 of color two.

Let $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathcal{F} = \{M_1, M_2\}$.

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Open problems

One could maybe try to generalize the results to 3-partitions (using 3×3 matrices), but this is maybe difficult and I do not have any immediate applications.

But it would be interesting to see if there are any other problems that can be solved using the above dichotomy.

Or one could try to prove the same dichotomy, where we do not require that if a matrix belongs to $\mathcal{F}$ then all multiples of that matrix is also allowed to be used in the digraph.

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The End

Any questions?