Flows, coloring, and homology

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Coloring and graphs on surfaces

Theorem (The Four Color Theorem)

Every planar graph is 4-colorable.

Theorem (Heawood's formula)

Every graph drawn on a surface of Euler genus g is

$$\left\lfloor \frac{7+\sqrt{24g+1}}{2} \right\rfloor\text{-colorable.}$$

Theorem (Grötzsch theorem)

Every planar triangle-free graph is 3-colorable.

The key question

Problem

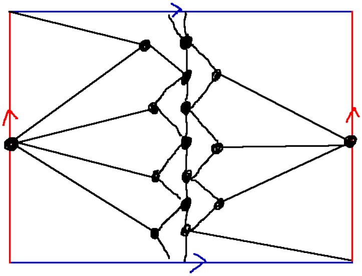
Can the Four Color Theorem and Grötzsch theorem be generalized to other surfaces?

Perhaps with a few exceptional graphs?

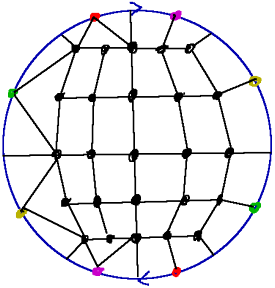
Answer

No.

Non-3-colorable triangle-free graphs

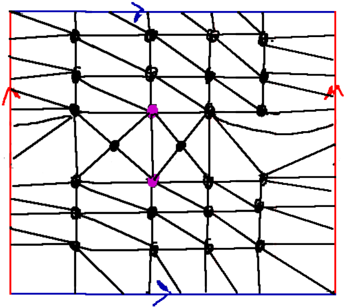


Mycielsky graphs of odd cycles

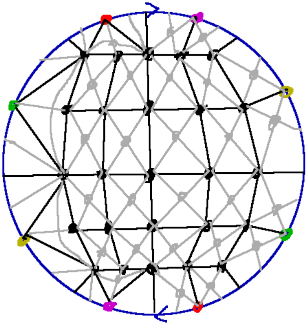


Non-bipartite quadrangulations
of the projective plane

Non-4-colorable graphs



Triangulations with two adjacent vertices of odd degree.



Triangulations of non-3-colorable quadrangulations.

The key question, revisited

For a surface Σ, girth γ, number of colors k:

Problem (Structural characterization)

How do the minimal non-k-colorable graphs of girth at least γ drawn on Σ look like?

Problem (Algorithms)

Given a graph G of girth at least γ drawn on Σ, can we decide whether G is k-colorable in polynomial time?

Finitely many obstructions

For a surface Σ, girth γ, number of colors k:

Theorem (Thomassen'93, Gallai'63, Thomassen'03)

If

- $k \geq 5$, *or*
- $\gamma \geq 4$ *and* $k = 4$, *or*
- $\gamma \geq 5$ *and* $k = 3$,

then there are only finitely many minimal non-k-colorable graphs of girth at least γ drawn on Σ.

Corollary

In all these situations, colorability can be tested by checking the presence of finitely many obstructions.

Coloring graphs on surfaces

Critical graphs/complexity of coloring:

colors \ girth	3	4	≥ 5
3	∞/NPC	∞ but structured/P	finite/P
4	∞/open	finitely many/P	
≥ 5	finitely many/P		

Minimal non-3-colorable triangle-free graphs

Definition

A graph G drawn on a surface is an s-near-quadrangulation if

$$\sum_{f \in F(G), |f| \neq 4} |f| \leq s.$$

Theorem (D., Král', Thomas'09)

For every surface Σ, there exists $s = O(g(\Sigma))$ such that every minimal non-3-colorable triangle-free graph drawn on Σ without non-contractible 4-cycles is an s-near-quadrangulation.

Reducing to a near-quadrangulation

Theorem (D., Král', Thomas'09)

For every surface Σ, there exists s and a linear-time algorithm that given a triangle-free graph G drawn on Σ without non-contractible 4-cycles either

- *correctly decides that G is 3-colorable, or*
- *returns a subgraph $H \subseteq G$ such that*
  - *every 3-coloring of H extends to a 3-coloring of G, and*
  - *H is an s-near-quadrangulation.*

Coloring near-quadrangulations

Theorem (D., Král', Thomas'09)

For every surface Σ and every s, there exists a linear-time algorithm that given an s-near-quadrangulation H of Σ either

- *finds a 3-coloring of H, or*
- *correctly decides that H is not 3-colorable.*

Corollary (D., Král', Thomas'09)

For every surface Σ, there exists s and a linear-time algorithm that given a triangle-free graph G drawn on Σ correctly decides whether G is 3-colorable.

Problem

Does there exist a simple algorithm to 3-color near-quadrangulations?

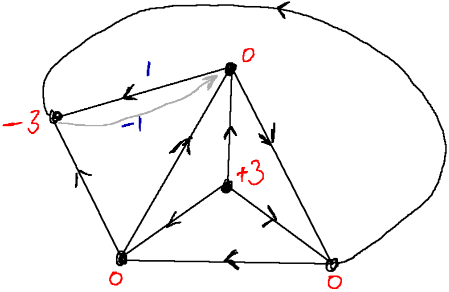
Flows

Definition

A flow with boundary $d : V(G) \to \mathbb{Z}$ is a function ψ from directed edges of G to $\mathbb{Z}$ such that

1. $\psi(u,v) = -\psi(v,u)$ for every $uv \in E(G)$, and
2. for every $u \in V(G)$,
$$\sum_{uv \in E(G)} \psi(u,v) = d(u).$$

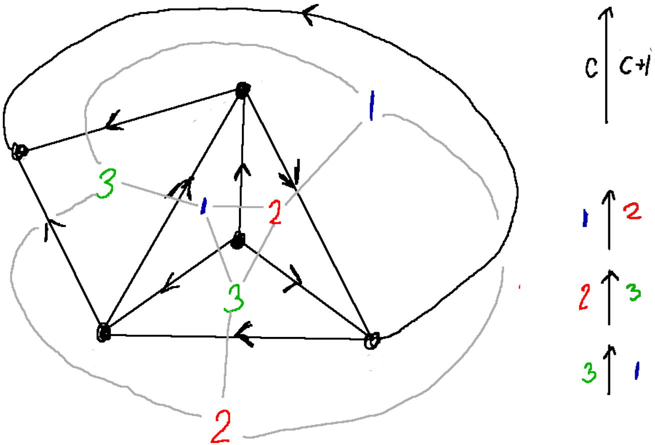
$(\leq k)$-flow: $|\psi(u,v)| \leq k$, nowhere-zero: $\psi(u,v) \neq 0$.



Planar case: Flow-coloring duality

Theorem (Tutte'54)

A plane graph H is 3-colorable if and only if its dual graph G has a nowhere-zero (≤ 1)-flow with boundary divisible by 3.



A 3-coloring algorithm

Definition

A function $d : V(G) \to \mathbb{Z}$ is a plausible boundary if

- $\sum_{v \in V(G)} d(v) = 0$ and

for each $v \in V(G)$,

- $3 | d(v)$,
- $d(v) \equiv \deg v \pmod 2$, and
- $|d(v)| \leq \deg v$.

Algorithm (3-colorability of a plane graph H)

For every plausible boundary d for the dual G of H:

- *If there exists a nowhere-zero (≤ 1)-flow in G with boundary d, return true.*

Otherwise, return false.

Number of choices for d?

Possible boundary values b at a vertex of degree k:

- $3 | b$,
- $b \equiv k \pmod 2$, and
- $|b| \leq k$.

k	3	4	5	6	7	8	9
choices	± 3	0	± 3	$0, \pm 6$	± 3	$0, \pm 6$	$\pm 3, \pm 9$
number of choices	2	1	2	3	2	3	4

Corollary

For an s-near-quadrangulation H, the number $q(H)$ of plausible boundaries for the dual of H is bounded by a function of s.

Applications

For plane *s*-near-quadrangulations:

- Precoloring extension from a connected subgraph (D., Lidický'15)
- Precoloring extension from a subgraph with two components (D., Pekárek'21).

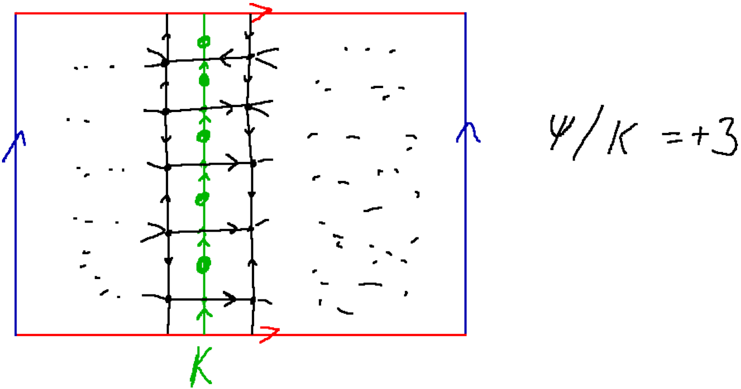
For triangle-free graphs drawn on the torus:

- Practical linear-time algorithm for 3-colorability (D., Pekárek'21)
- Implemented by Urmanov'22.

Flow over cycles

For a cycle K in a graph H and flow ψ in its dual, let

$$\psi/K = \sum_{e \in E(K)} \psi(e^{\star})$$



Generalized version of Tutte's duality

Lemma

The following claims are equivalent for a graph H drawn on an orientable surface:

- *the graph H is 3-colorable*
- *there exists a nowhere-zero (≤ 1)-flow ψ in the dual G of H such that*

$$3|\psi/K$$

  for every cycle K in H.

Observation

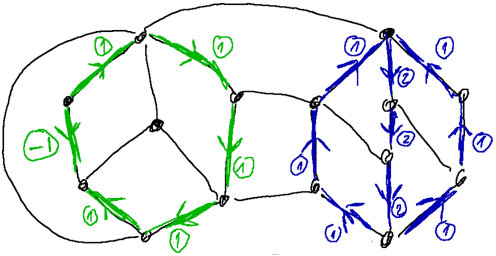
The condition holds for contractible cycles iff the boundary of ψ is divisible by 3.

Integral cycles

Definition

A 1-cycle in H is a flow with zero boundary. For a flow ψ in the dual G of H,

$$\psi/K = \sum_{e \in E(H)} K(e) \cdot \psi(e^\star).$$



Linearity

Observation

If K_1 and K_2 are 1-cycles and n is an integer, then

- *$K_1 + K_2$ is a 1-cycle and*

$$\psi/(K_1 + K_2) = \psi/K_1 + \psi/K_2.$$

- *nK_1 is a 1-cycle and*

$$\psi/(nK_1) = n \cdot \psi/K_1.$$

The space of cycles

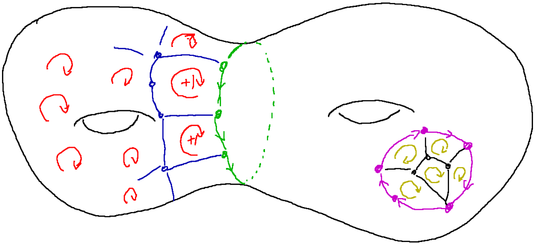
Definition

For a graph H drawn on an orientable surface, let

- $\mathcal{C}(H)$ be the set of all 1-cycles, and
- $\mathcal{B}(H) \subseteq \mathcal{C}(H)$ the set of linear combinations (with integer coefficients) of face boundaries of H, called 1-boundaries.

Observation

A cycle K in H belongs to $\mathcal{B}(H)$ iff K separates the surface. In particular, all contractible cycles belong to $\mathcal{B}(H)$.



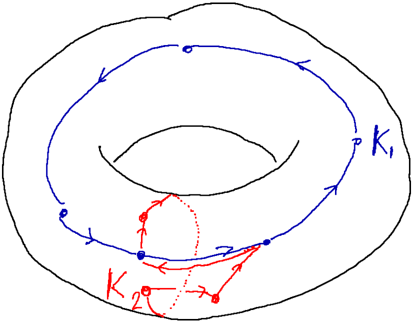
Generators

For a graph H on an orientable surface of Euler genus g:

Lemma

There exist non-contractible cycles $K_1, \ldots, K_g$ in H (generators of the homology group of H) such that

$$\mathcal{C}(H) = \left\{ B + \sum_{i=1}^{g} n_i K_i : n_1, \ldots, n_g \in \mathbb{Z}, B \in \mathcal{B}(H) \right\}.$$



General surfaces, fewer cycles

Lemma

The following claims are equivalent for a graph H drawn on an orientable surface and generators $K_1, \ldots, K_g$ of its homology group:

- *the graph H is 3-colorable*
- *there exists a nowhere-zero (≤ 1)-flow ψ with boundary divisible by 3 in the dual G of H such that*

$$3|\psi/K_i$$

for $i = 1, \ldots, g$.

Algorithm for general surfaces

A function $r : [g] \to \mathbb{Z}$ is plausible if for each i, $3|r(i)$, $|K_i| \equiv r(i) \pmod 2$, and $|r(i)| \leq |K_i|$.

Algorithm (3-colorability of a graph H on orientable surface)

- *Find generators $K_1, \ldots, K_g$ of the homology group of H.*
- *For*
  - *every plausible boundary d for the dual G of H and*
  - *every plausible $r : [g] \to \mathbb{Z}$*

  check:
  - *If there exists a nowhere-zero (≤ 1)-flow ψ in G with boundary d such that $\psi / K_i = r(i)$ for $i = 1, \ldots, g$, return true.*

Otherwise, return false.

- $O(q(H) \cdot n^g)$ choices.
- How to test the existence of the flow?
  - Idea from Chambers, Erickson, Nayyeri'10, Venkatesan'83.

Finding the flow

Problem

Given

- *a plausible boundary d in the dual G of H and*
- *a plausible function $r : [g] \to \mathbb{Z}$,*

decide efficiently whether there exists a nowhere-zero (≤ 1)-flow ψ in G with boundary d such that $\psi / K_i = r(i)$ for $i = 1, \ldots, g$.

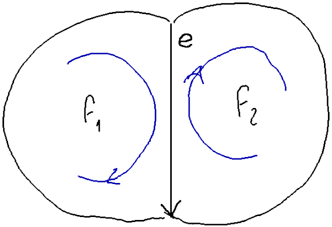
- A flow ψ_0 with these properties always exists (sum of paths and dual generators).
- $\psi - \psi_0$ has zero boundary (it is a 1-cycle) and $\psi / K_i = 0$ for $i = 1, \ldots, g$.
- Equivalently, $\psi - \psi_0 \in \mathcal{B}(G)$!

Finding the flow, simplified

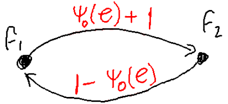
Problem

Given a graph G on a surface and a flow ψ_0 in G, decide efficiently whether there exists a 1-boundary $R \in \mathcal{B}(G)$ such that $\psi_0 + R$ is a nowhere-zero (≤ 1)-flow.

$$R = \sum_{f \in F(G)} a_f \partial f$$



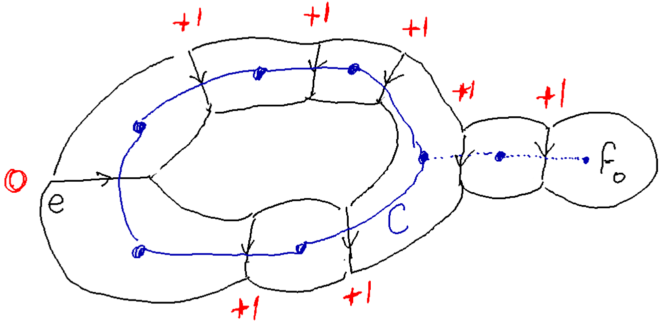
$$-1 \leq \psi_0(e) + a_{f_1} - a_{f_2} \leq 1$$
$$a_{f_1} \leq a_{f_2} + (1 - \psi_0(e))$$
$$a_{f_2} \leq a_{f_1} + (\psi_0(e) + 1)$$



Solution: $a_f = d_\ell(f_0, f)$.

Is it nowhere-zero?

Suppose $(\psi_0 + R)(e) = 0$:



$$\psi_0 / C = (\psi_0 + R)/C \not\equiv |C| \pmod 2$$

- Implies that $\psi_0 + R$ is not a nowhere-zero (≤ 1)-flow for any $R \in \mathcal{B}(G)$.
- Will not happen because of the parity conditions in plausibility.

Summary

The following are equivalent for plausible d and r:

- There exists a nowhere-zero (≤ 1)-flow ψ with boundary d such that $\psi / K_i = r(i)$ for each i.
- The corresponding auxiliary graph does not contain a negative length cycle.
- For every cycle $C = B + \sum_i n_i K_i$ with $B \in \mathcal{B}(H)$,

$$\langle B, d \rangle + \sum_i n_i r(i) \leq |C|.$$

Polytope of realizable flows

For a plausible boundary d in G, let

$$u_d(n_1,\dots,n_g) = \min_{B\in\mathcal{B}(H)} \left|B + \sum_i n_i K_i\right| - \langle B, d\rangle.$$

$$\mathcal{P}_d = \left\{ x \in \mathbb{R}^g : \sum_{i=1}^{g} n(i)\cdot x(i) \le u_d(n) \text{ for every } n \in \mathbb{Z}^g \right\}.$$

Observation

The following claims are equivalent:

1. *There exists a nowhere-zero* (≤ 1)*-flow* ψ *with boundary* d *such that* $3|\psi/K_i$ *for each* i*.*
2. $\mathcal{P}_d$ *contains a point* x *with integer coordinates such that*

$$x(i) \equiv 3(|K_i| \bmod 2) \pmod 6 \text{ for } i = 1,\dots,g.$$

3. $(\mathcal{P}_d - 3(|K_\star| \bmod 2))/6$ *contains a point with integer coordinates.*

An improved algorithm

To decide whether a graph H on an orientable surface of Euler genus g is 3-colorable:

Algorithm (Bang, D., Heath, Lidický'22)

For every plausible boundary d for the dual G of H,

- *if* $(\mathcal{P}_d - 3(|K_\star| \bmod 2))/6$ *contains a point with integer coordinates, return true.*

Otherwise, return false.

Time complexity:

- $q(H) \cdot O(|H|^2 \text{polylog}|H|)$
- Worse than $O(|H|)$ algorithm of D., Král', Thomas.
- But easy to implement.

Variations

- Include choice of d in the polytope.
  - Removes the dependency on the lengths of the faces, but increases the dimension.
- Non-orientable surfaces (D., Moore, Sereni).
- Homomorphism to C_{2k+1} instead of 3-coloring.
- More generally, circular $(a : b)$-coloring for a odd.
  - a even fails (0 cannot have different parity from all allowed flow values).
- Any number of precolored vertices.

Edgewidth of non-3-colorable quadrangulations

Edgewidth: The length of the shortest non-contractible cycle.

Theorem (Hutchinson'94)

If H is a quadrangulation of an orientable surface of Euler genus g and the edgewidth of H is at least $\exp(\Theta(g))$, then H is 3-colorable.

Theorem (D., Král', Thomas'09)

Edgewidth $\Theta(g^3)$ suffices, even for triangle-free graphs.

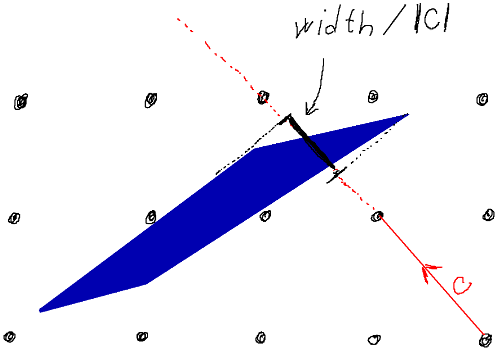
Polytope width

Definition

The width of a polytope $\mathcal{P} \subset \mathbb{R}^g$ is the infimum of

$$\max_{x \in \mathcal{P}} \langle c, x \rangle - \min_{x \in \mathcal{P}} \langle c, x \rangle$$

over $c \in \mathbb{Z}^g \setminus \{(0, \ldots, 0)\}$.



Polytopes without integer points

Theorem (Kannan, Lovász'88, Rudelson'00)

If a bounded polytope in $\mathbb{R}^g$ does not contain an integer point, then it has width $O(g^{4/3})$.

Conjecture (Banaszczyk et al.'99)

If a bounded polytope in $\mathbb{R}^g$ does not contain an integer point, then it has width $O(g \log g)$.

Width of the polytope of realizable circulations

Lemma (Bang, D., Heath, Lidický'22)

If H is an s-near-quadrangulation and the dual G of H has a (≤ 1)-flow with boundary d, then the width of $\mathcal{P}_d$ is $\Omega(ew(H)) - O(s)$.

Recall:

Theorem (D., Král', Thomas'09)

For every surface Σ, there exists $s = O(g(\Sigma))$ such that every minimal non-3-colorable triangle-free graph drawn on Σ without non-contractible 4-cycles is an s-near-quadrangulation.

Corollary

If H is a triangle-free graph drawn on an orientable surface of Euler genus g and the edgewidth of H is at least $\Theta(g^{4/3})$, then H is 3-colorable.

Quadrangulations

Observation

If H is a quadrangulation, then $\mathcal{P}$ is centrally symmetric.

Theorem (Banaszczyk et al.'96)

If a centrally symmetric bounded polytope in $\mathbb{R}^g$ does not contain an integer point, then it has width $O(g \log g)$.

Corollary

If H is a quadrangulation of an orientable surface of Euler genus g and the edgewidth of H is at least $\Theta(g \log g)$, then H is 3-colorable.

Conclusions

Summary

One part of D., Kráľ, Thomas'09 argument (dealing with near-quadrangulations) can be simplified and better understood via nowhere-zero flows.

Problem

What about the other part:

- *A practical algorithm to reduce the problem of 3-colorability of a triangle-free graph H to a near-quadrangulation $H_0 \subseteq H$?*
- *Tight bounds on $q(H_0)$?*

Theorem (D., Pekárek)

If H is drawn on the torus, then $q(H_0) \leq 16$.

Conclusions

Summary

The concept of homology is useful even if you have no idea what it is.