Vizing's Conjecture: different approaches

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Basic definitions

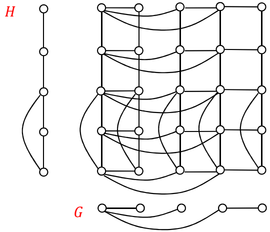
$G = (V(G), E(G))$

- Open neighborhood $N_G(v)$: set of neighbors of a vertex v; $N_G(S) = \bigcup_{v \in S} N_G(v)$.
- Closed neighborhood: $N_G[v] = N_G(v) \cup \{v\}$; $N_G[S] = \bigcup_{v \in S} N_G[v]$.
- Each vertex dominates itself and its neighbors.
  Dominating set $D \subseteq V(G)$: $N_G[D] = V(G)$.
- Domination number: minimum cardinality $\rightarrow \gamma(G)$.
- Packing $S \subseteq V(G)$: the closed neighborhoods of the vertices are pairwise disjoint.
- Packing number: minimum cardinality of a packing $\rightarrow \rho(G)$.

$$\rho(G) \leq \gamma(G).$$

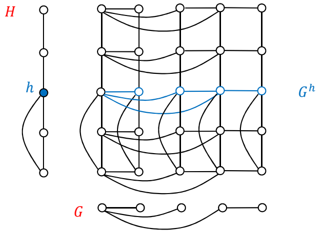
Cartesian product

- Cartesian product of two graphs:
  - $V(G \,\square\, H) = V(G) \times V(H)$;
  - Two vertices, (g, h) and (g', h') are adjacent if
    $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$.



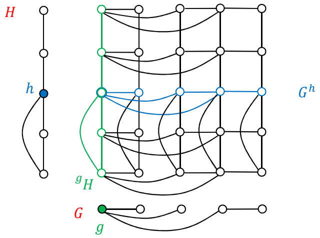
Cartesian product

- G-fiber in $G \,\square\, H$:
  For a fixed $h \in V(H)$: $G^h = \{(x, h) : x \in V(G)\}$.



Cartesian product

- *H*-fiber in $G \,\Box\, H$:
  For a fixed $h \in V(H)$: $\quad {}^{g}H = \{(g, y) : y \in V(H)\}$.



Vizing's Conjecture

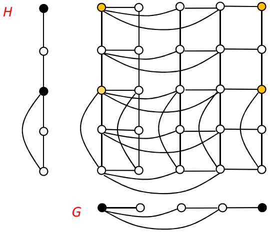
Vizing's Conjecture (1968)

For every two graphs G and H, it holds that

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H).$$

$\gamma(G \,\square\, H)$

- The Cartesian product of two dominating sets is typically not enough to dominate $G \,\square\, H$.

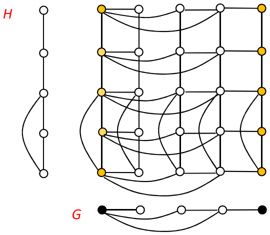


$\gamma(G \,\square\, H)$

- For every two graphs:

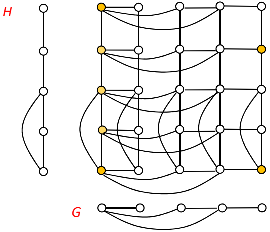
$$\gamma(G \,\square\, H) \le \min\{n_G \cdot \gamma(H), \gamma(G) \cdot n_H\}.$$

- Here, each vertex is dominated horizontally (i.e., inside a G-fiber):



$\gamma(G \,\square\, H)$

- A subset is also a dominating set (now, some vertices are dominated only vertically):



Vizing's Conjecture: some related facts

Vizing's Conjecture (1968)

For every two graphs G and H, it holds that

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H).$$

- If G' is a spanning subgraph of G such that $\gamma(G') = \gamma(G)$, then for every graph H,

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H) \quad \text{implies} \quad \gamma(G' \,\square\, H) \geq \gamma(G')\gamma(H).$$

Proof: The domination number of $G \,\square\, H$ cannot drop while deleting edges and obtain $G' \,\square\, H$.

$$\gamma(G' \,\square\, H) \geq \gamma(G \,\square\, H) \geq \gamma(G)\gamma(H) = \gamma(G')\gamma(H).$$

Vizing's Conjecture: related observations

Vizing's Conjecture (1968)

For every two graphs G and H, it holds that

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H).$$

- If G' is a spanning subgraph of G such that $\gamma(G') = \gamma(G)$, then for every graph H,

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H) \quad \text{implies} \quad \gamma(G' \,\square\, H) \geq \gamma(G')\gamma(H).$$

- $\min\{n_G, n_H\} \leq \gamma(G \,\square\, H) \leq \min\{\gamma(G) \cdot n_H, \gamma(H) \cdot n_G\}$
[El-Zahar, Pareek, 1991; Vizing, 1963]

Graphs satisfying Vizing's Conjecture

We say that a graph G satisfies Vizing's Conjecture if $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$ is true for every graph H.

Graph classes which are proved to satisfy Vizing's Conjecture:

- chordal graphs [Aharoni, Szabó, 2009]
- graphs with domination number $\gamma \leq 3$ [Barcalkin, German, 1979; Sun, 2004; Brešar, 2015]
- BG-graphs (e.g., trees) [Barcalkin, German, 1979]
- Type $\mathcal{X}$ graphs [Hartnell, Rall, 1994]

Weaker lower bounds on $\gamma(G \,\square\, H)$

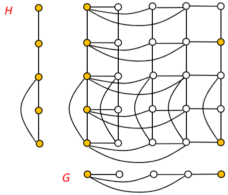
For every two graphs:

- $\gamma(G \,\square\, H) \geq \frac{1}{2}\gamma(G)\gamma(H)$ [Clark, Suen, 2000]
- $\gamma(G \,\square\, H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \max\{\gamma(G), \gamma(H)\}$ [Zerbib, 2019]

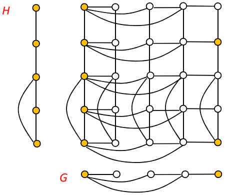
For a claw-free graph G and an arbitrary H:

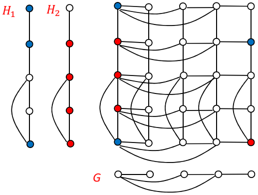
- $\gamma(G \,\square\, H) \geq \frac{3}{4}\gamma(G)\gamma(H)$ [Brešar, Henning, 2020]

Projection



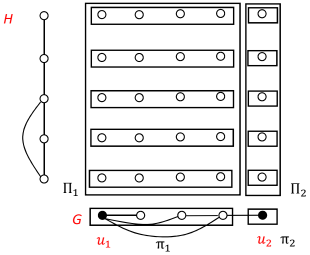
Projection





Partition classes, blocks and cells

- Let $D_G = \{u_1, \ldots, u_k\}$ a minimum dominating set in G. ($k = \gamma(G)$)
- Choose a partition $\pi_1, \ldots \pi_k$ of $V(G)$ such that $u_i \in \pi_i$ and $\pi_i \subseteq N[u_i]$ for every i.
- Block $\Pi_i = \pi_i \times H$.
- Cell $C_i^h = \pi_i \times \{h\}$ for $h \in V(H)$ and $1 \leq i \leq k$.



Decomposable graphs

- G is decomposable if there is a partition $\pi_1, \ldots \pi_k$ of $V(G)$ s.t. each π_i induces a clique. ($k = \gamma(G)$)

Lemma: Let G be a decomposable graph with a partition $\pi_1, \ldots, \pi_k$ and $D \subseteq V(G)$. If $\pi_1, \ldots, \pi_\ell$ are the classes which are disjoint from D and $D \cap \bigcup_{i=\ell+1}^{\ell+t} \pi_i$ externally dominates $\bigcup_{i=1}^{\ell} \pi_i \Rightarrow$

$$|D \cap \bigcup_{i=\ell+1}^{\ell+t} \pi_i| \geq \ell + t.$$

Theorem (Barcalkin, German, 1979)

Every decomposable graph satisfies Vizing's Conjecture.

Proof: Project the vertices of a dominating set D onto $k = \gamma(G)$ copies of H. If a cell C_i^h contains some vertices from D, then one of them is projected to the vertex h in the i^{th} copy H_i. If a vertex is not dominated in a copy H_j, we can use the lemma and project a surplus vertex from another cell C_s^h to H_j.

Consequences

Theorem (Barcalkin, German, 1979)

Every decomposable graph satisfies Vizing's Conjecture.

- If G' is a spanning subgraph of a decomposable graph G such that $\gamma(G') = \gamma(G)$ (G' is a BG-graph), then G' satisfies Vizing's Conjecture.
- The following graphs satisfy Vizing's Conjecture:
  - graphs with $\gamma(G) \leq 2$,
  - graphs with $\gamma(G) = \rho(G)$ (e.g., trees).

Graphs of Type $\mathcal{X}$

- G is of Type $\mathcal{X}$ if $V(G)$ can be partitioned into $SC \cup BC \cup C \cup S$ such that
  - SC: induces a clique where every vertex has a neighbor outside SC;
  - $BC = B_1 \cup \cdots \cup B_t$ where every B_i is a clique with a vertex b_i, $N[b_i] = B_i$;
  - $C = C_1 \cup \cdots C_m$ where each C_i is a clique;
  - $S = S_1 \cup \cdots \cup S_k$ where every S_i is a "star-like" graph with a vertex s_i, $N[s_i] = S_i$, and S_i satisfies some further conditions;
  - There are no edges between C and S; and
  - $\gamma(G) = 1 + t + m + k$.

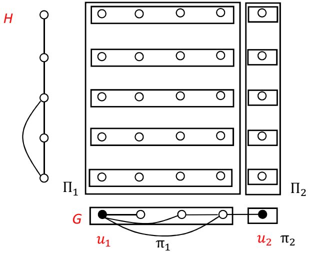
Theorem (Hartnell, Rall, 1995)

Every graph of Type $\mathcal{X}$ satisfies Vizing's Conjecture.

Consequences: Vizing's Conjecture is true for all graphs with $\gamma(G) = \rho(G) + 1$; and for spanning subgraphs of Type $\mathcal{X}$ graphs if the domination number is the same.

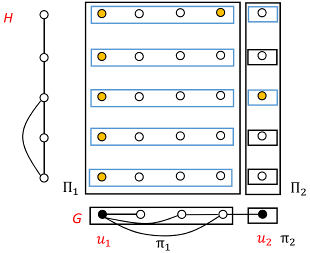
Double projection method

- $D_G = \{u_1, \ldots, u_k\}$: a minimum dominating set in G $\quad (k = \gamma(G))$;
- $\pi_1, \ldots, \pi_k$: a partition of $V(G)$ as before;
- blocks Π_i, cells C_i^h;
- Fix a (minimum) dominating set D in $G \Box H$.



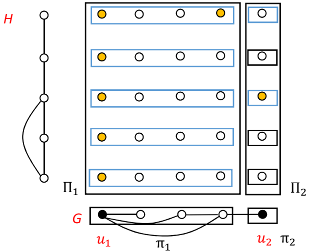
Double projection method

- $D_G = \{u_1, \ldots, u_k\}$: a minimum dominating set in G $(k = \gamma(G))$;
- $\pi_1, \ldots, \pi_k$: a partition of $V(G)$ as before;
- blocks Π_i, cells C_i^h;
- Fix a minimum dominating set D in $G \square H$.
- A cell is blue if its every vertex is dominated horizontally.



Double projection method

- B: # of blue cells in $G \,\square\, H$;
- B_i: # of blue cells in block Π_i;
- B_h: # of blue cells in fiber G^h.
- $B = \sum_{i=1}^{k} B_i = \sum_{h \in V(H)} B_h$



Double projection method

- B: # of blue cells in $G \,\Box\, H$;
- B_i: # of blue cells in block Π_i; B_h: # of blue cells in fiber G^h.
- $B = \sum_{i=1}^{k} B_i = \sum_{h \in V(H)} B_h$.
- In a fiber G^h: $B_h \leq |D \cap G^h| \;\Rightarrow\; B \leq |D|$.
- In a block Π_i: $B_i \geq \gamma(H) - |D \cap \Pi_i| \;\Rightarrow\; B \geq k\gamma(H) - |D|$.

$$|D| \geq B \geq k\gamma(H) - |D|$$

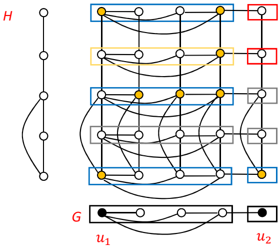
Theorem (Clark, Suen, 2000)

It holds for every two graphs G and H:

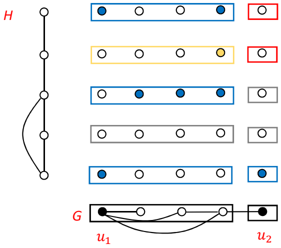
$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma(H).$$

More colors

- A cell C_i^h is blue: $C_i^h \cap D \neq \emptyset$ and $\forall$ vertex of C_i^h is dominated horizontally.
- A cell C_i^h is yellow: if $C_i^h \cap D \neq \emptyset$ and not every vertex of C_i^h is dominated horizontally.
- A cell C_i^h is white: if $C_i^h \cap D = \emptyset$ and $\exists$ a vertex which is dominated vertically.
- A cell C_i^h is red: if $C_i^h \cap D = \emptyset$ and no vertex is dominated vertically.

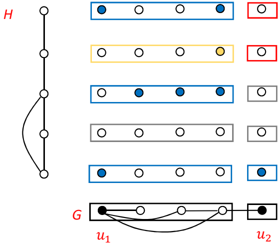


- B: # of blue cells in total;
  B_i: # of blue cells in the block Π_i;
  B_h: # of blue cells in the fiber G^h;
- Similarly, for the number of yellow, white, and red cells: $Y, Y_i, Y_h, W, \ldots$
- The vertices of the dominating set D are colored with blue and yellow according to the color of the cell.
- b: # of blue vertices in total; y: # of yellow vertices in total.



Lemma (Brešar, Hartnell, Henning, Kuenzel, Rall, 2021)

- $B + Y + R \geq \gamma(G)\gamma(H)$
- $b + y \geq B + R$



Alternative proof for the Clark-Suen theorem:

$$\gamma(G)\gamma(H) \leq B + Y + R \leq= b + y + Y$$
$$= |D| + Y \leq 2\gamma(G \,\Box\, H).$$

A strengthening of the Clark-Suen theorem:

Theorem (Brešar, Hartnell, Henning, Kuenzel, Rall, 2021)

$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma_t(H).$$

A different domination in Cartesian products

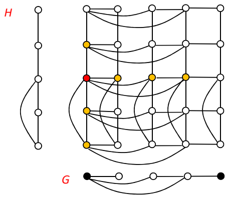
Definition

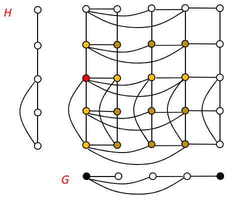
Let G and H be graphs and C a fixed minimum dominating set in G. We say that a vertex (x, y) C-dominates (x', y') in $G \Box H$ if

- $(x, y) = (x', y')$, or
- (x, y) and (x', y') are adjacent in $G \Box H$, or
- $x \in C$ and both $xx' \in E(G)$ and $yy' \in E(H)$ hold.

The smallest size of a set D that C-dominates $G \Box H$ is the C-domination number $\gamma(C, G \Box H)$ of the Cartesian product.

You may also say that, if $x \in C$, then (x, y) dominates $N_G[x] \times N_H[y]$ in $G \Box H$.


H
G


H
G

Some facts and questions

- Fact: If C and D are minimum dominating sets in G and H, resp., then $C \times D$ is a C-dominating set in $G \,\Box\, H$.

$$\gamma(C, G \,\Box\, H) \leq \gamma(G)\gamma(H).$$

- Fact: $\gamma(G \,\Box\, H) \geq \gamma(C, G \,\Box\, H)$.
- Question: Is it true that for every line graph G there exists a minimum dominating set C such that for every graph H

$$\gamma(C, G \,\Box\, H) = \gamma(G)\gamma(H) \text{ holds ?}$$

  If the answer is YES, then line graphs satisfy Vizing's Conjecture.
- The same question relates to other graph classes.

Thank you for your attention!